Gompertz uniform {exponential} three parameter distribution and Application

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Abstract:

This paper introduces the Gompertz uniform exponential (GU{EXP}) three parameter distribution. Some mathematical properties of this distribution are studied. Density distribution, Reliability and hazard rate functions are obtained. The ordinary moments, quintile function, mean residual life, Renyi entropy are given. Four methods of estimation of the (GU{EXP}) distribution based on complete sampling and the MLE estimates based on censoring type | and || are given. Squared Bias and variances of the estimates via a Mont Carlo simulation study are computed. We introduced also a real data analysis.

Keyword: Gompertz distribution, T-X families, Quantile function, Renyi Entropy

1- Introduction

The Gompertz distribution was introduced by BENJAMIN GOMPERTZ (1825). It has been used to describe human mortality and establish actuarial tables[8]. AHUJA (1967) introduced The Generalized Gompertz-Verhulst Family of Distributions [15]. GARG (1970) discussed The maximum likelihood estimates for the parameters of the Gompertz distribution [23]. Gordon (1990) introduced Maximum likelihood estimation for mixtures of two gompertz distributions when censoring occurs [24]. FRANSES (1994) proposed a simple Gompertz curve-fitting procedure [26]. Chen (1997) focused on exact confidence

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interval for the parameter c and exact joint confidence region for the parameters of the Gompertz distribution [31]. Wu (2003) Point and interval estimations for the Gompertz distribution under progressive type-II censoring [17]. Wu (2004) proposed unweighted and weighted least squares estimates for parameters of the Gompertz distribution under the complete data and the first failure-censored data [16]. El-Gohary (2013) dealed with a new generalization of the exponential, Gompertz, and generalized exponential distributions. This distribution is called the generalized Gompertz distribution[5]. Srivastava (2013) obtained the tables and graphs of critical values of Kolmogorov-Smirnov (KS) test, and Q-Q test for Gompertz model with two unknown parameters [4]. Ghitany (2014) considered a progressively Type-II censored sample from the two-parameter Gompertz distribution [22]. Torres (2014) proposed a Nonlinear least squares procedures for estimating the parameters of the shifted Gompertz distribution [10]. Abu-Zinadah (2014) discussed some characterizations of the exponentiated Gompertz distribution [12] Jafari (2014) introduce a new four-parameter generalized version of the Gompertz model which is called Beta-Gompertz (BG) distribution [7]. Abdul-Moniem (2015) introduce a new distribution called transmuted Gompertz distribution (TGD) [14]. Ahmed (2015) studied the estimation of parameters of a three-parameter generalized Gompertz distribution based on progressively type-II right censored sample [9]. Alizadeh (2016) studied general mathematical properties of a new generator of continuous distributions with three extra parameters called the exponentiated Gompertz generated (EGG) family [19]. Bakouch (2017) introduced A new weighted version of the Gompertz distribution [13]. Alizadeh (2017) introduce and study some general mathematical properties of a Gompertz-G generator [21]. Eliwa (2017) introduced a new model based on exponentiated generalized Weibull-Gompertz distribution [6]. Oguntunde (2018) derived The gompertz inverse exponential using the gompertz generalized family of distributions [25] Roozegar (2017) introduces a five - parameter life time model called the McDonald Gompertz [27]. Ieren (2019) introduced a three-parameter probability distribution which gives another extension of the Gompertz distribution [29]. Al-Noor (2020) introduced a new compound distribution termed as Marshall Olkin Exponential Gompertz (MOEGo). [20]. Khaleel (2020) introduced the Gompertz flexible Weibull distribution as an extension of theflexible Weibull distribution [18]. Chipepaa (2021) developed a new generalized family of the Gompertz-G distribution, namely, the Marshall-Olkin-Gompertz-Gdistribution [11]. Ogunde (2021) introduced Gompertz Gumbel II (GG II) distribution which generalizes the Gumbel II distribution [2]. Gui (2021) discuss the estimation of the parameters for Gompertz distribution and prediction using general progressive Type-II censoring [30].

2-A new distribution derived from T-X{Y} Family: Gompertz uniform exponential (GU{EXP}).

Alizadeh et al. (2017) developed Gompertz-G family of distributions by using the T-X{Y} family defined as:

The CDF of the T-X family is defined as:

$$G(x) = \int_{a}^{w_y(F(x))} r(t)dt = R\{w[F(x)]\}$$

Where R is the CDF of T. then the PDF g(□) is given as:

$$\text{g(x)} = \text{r}\{\text{w[F(x)]}\}\{\tfrac{d}{dx} w_y[F(x)]\}$$

We can take the $w_y(.)$ function to be the quantile function of Y, then:

$$G(x) = \int_{a}^{Q_y(F(x))} r(t)dt$$

$$-\infty < a \le T \le b < \infty$$

Let r(.) and R(.) be the PDF and the CDF of the Gompertz distribution, where:

r(t) = $\lambda e^{t} e^{-\lambda(e^{t}-1)}, t > 0.\alpha, \lambda > 0$

we derive a new T-X{Y} family using the quantile function of the exponential distribution, where the random variable y has the standard exponential distribution the PDF is:

$$p(y) = e^{-y}, 0 \le y \le \infty$$

With the CDF:

$$\mathrm{P}(y) = 1 - e^{-y}, 0 \le y \le \infty$$

The quantile function $Q_y(\theta)$ is the solution of the equation:

$$P(Q_y(\theta)) = \theta$$

Which:

$$Q_y(\theta) = -\ln(1-\theta)$$

Then:

$$Q_y(\mathrm{F(x)}) = -\ln(1-\mathrm{F(x)})$$

Then the Gompertz-G family three- parameter is given by:

$$G(x) = \int_{0}^{-\ln[1-F(x)]} \lambda e^{t} e^{-\lambda(e^{t}-1)} dt = 1 - e^{\lambda\{1-[1-F(x)]^{-1}\}} \ldots(1)$$

$$g(x)=\lambda f(x;v)[1-F(x;v)]^{-2}e^{\lambda\{1-[1-F(x;v)]^{-1}\}}...(2)$$

f(.) and F(.) are the PDF and the CDF of uniform distribution, where :

$$f(x)=\frac{1}{b-a},\mathrm{a}\le x\le b...(3)$$

$$F(x)=\frac{x-a}{b-a},\mathrm{a}\le x\le b...(4)$$

We obtain the PDF and the CDF of Gompertz uniform exponential (GU{EXP}) three parameter by putting (3) and (4) in (1) and (2) as:

$$\mathrm{g}(x)=\frac{\lambda}{b-x}[\frac{b-x}{b-a}]^{-1}e^{\lambda\{1-[\frac{b-x}{b-a}]^{-1}\}}$$

$$=\frac{\lambda}{(b-x)^2}e^{\lambda[\frac{a-x}{b-x}]}..........(5)$$

$$G(x)=1-e^{\lambda\{1-[\frac{b-x}{b-a}]^{-1}\}}$$

$$=1-e^{\lambda[\frac{a-x}{b-x}]}...................(6)$$

$$\lambda>0,\mathrm{a}\le\mathrm{x}\le\mathrm{b}$$

Where $\frac{a-x}{b-x}=1-[\frac{b-x}{b-a}]^{-1}$

By the PDF (5) and the CDF (6) we can write the survival function s(x) and the hazard function h(x) as:

$$s(x)=1-G(x)=1-1+e^{\lambda[\frac{a-x}{b-x}]}$$

$$\therefore\ s(x)=e^{\lambda[\frac{a-x}{b-x}]}$$

$$h(x)=\frac{g(x)}{s(x)}=\frac{\frac{\lambda}{b-a}\left[\frac{b-x}{b-a}\right]^{-2}e^{\lambda\left\{1-\left[\frac{b-x}{b-a}\right]^{-1}\right\}}}{e^{\lambda\left\{1-\left[\frac{b-x}{b-a}\right]^{-1}\right\}}}$$

$$\therefore\ h(x)=\frac{\lambda(b-a)}{(b-x)^2}$$

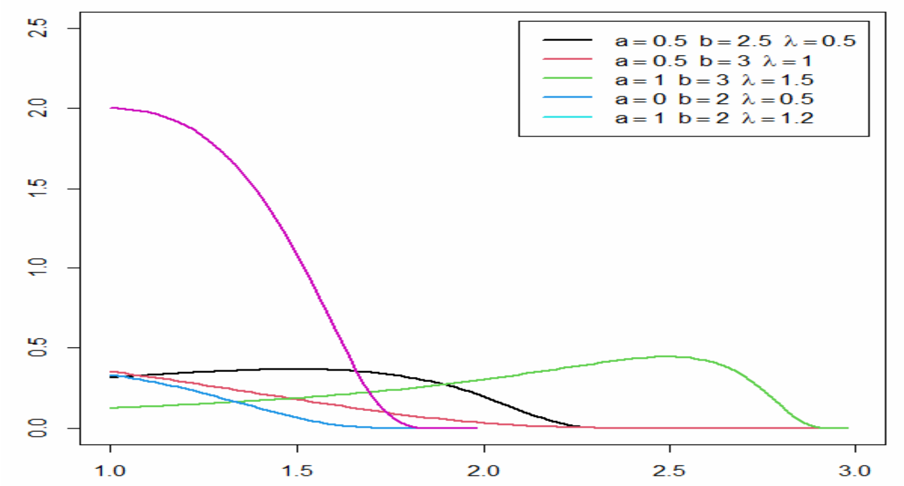

Figure 1. PDF of GUEXP distribution

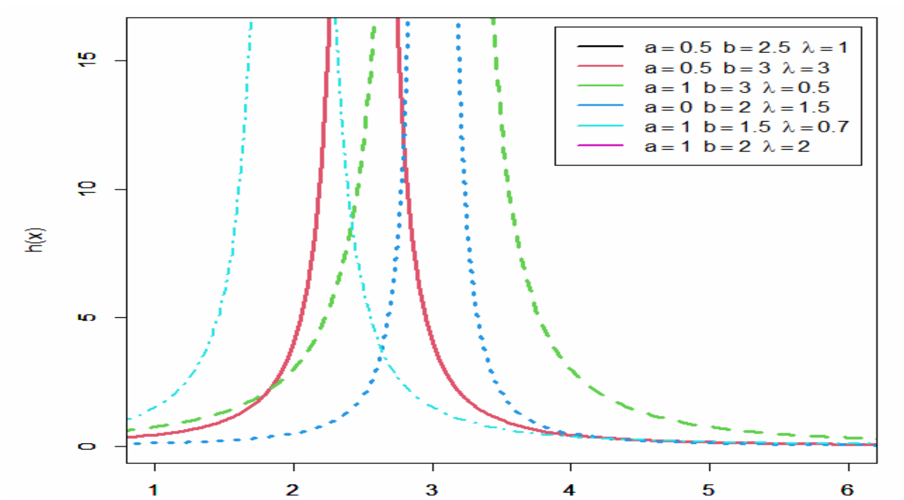

Figure 2.Hazard of GUEXP distribution

3-Quantile function

The Quantile function Q(p) or Q of the GU{EXP} distribution is the solution of the equation: G(Q)=P. then the quantile function Q at a vector p of percentiles is:

$$Q = b - \frac{\lambda(b-a)}{\lambda - \ln(1-p)} \quad ....(7)$$

We can obtain the generating function of the GU {EXP} distribution from equation (7) by writing x_u instead of Q and u instead of p, Where u is the uniform random variable (0,1) then:

$$\therefore x_u = b - \frac{\lambda(b-a)}{\lambda - \ln(1-u)}, \alpha, \lambda \succ 0$$

Then the skewness and kurtosis of GU{EXP} for different parameter values are given in the following table.

TABLE.1 Quantile

Parameters a, b, λ	0.25	0.5	0.75	0.125	0.375	0.625	0.875
a=0,b=3,λ=2	0.719205	1.732868	3.465736	0.333829	1.175009	2.452073	5.198604
a=0,b=3,λ=3	0.575364	1.386294	2.772589	0.267063	0.940007	1.961659	4.158883
a=0,b=3,λ=4	0.503444	1.213008	2.426015	0.23368	0.822506	1.716451	3.639023
a=0,b=3,λ=4.5	0.47947	1.155245	2.310491	0.222552	0.783339	1.634715	3.465736
a=0,b=3,λ=5	0.460291	1.109035	2.218071	0.21365	0.752006	1.569327	3.327106
a=0,b=3,λ=5.5	0.4446	1.071227	2.142455	0.206367	0.726369	1.515827	3.213682
a=0,b=3,λ=1	1.150728	2.772589	5.545177	0.534126	1.880015	3.923317	8.317766

4-Raw Moments

the r-th raw moment of the GU {EXP} variable x is

$$\mu_r' = \mathrm{E}(\mathrm{x}^r) = \int_a^b x^r \frac{\lambda}{(b-x)^2} e^{\lambda[\frac{a-x}{b-x}]} dx$$

The mean of x corresponds to r = 1. The mean, variance, skewness and kurtosis of the distribution for various values of the parameters are shown in Table 1. Table 1 indicates that if a, b and λ are fixed, the mean and variance of the GU {EXP} distribution

TABLE.2 moments and distribution properties

moments	a=0, b=3, λ=2	a=0, b=3, λ=4	a=0, b=3, λ=1	a=0, b=3, λ=3
mean	0.147902	0.063012	0.23132	0.091852
m2	0.111875	0.027288	0.220147	0.052551
m3	0.106614	0.017224	0.241369	0.041379
m4	0.114947	0.013782	0.285301	0.039135
M2	0.09	0.023317	0.166638	0.044114
CV	2.028371	2.423373	1.764713	2.286658
sk	2.349818	3.529305	1.666337	3.070308
kur	5.039699	15.47278	1.467544	10.55471

5-Renyi Entropy:

Entropy has been appeared by Alfred Renyi as a logarithmic measure of variation of uncertainty.

If we assume that the events $X=\{x_1, x_2, \ldots\ldots x_N\}$ have different probability $\{p_1, p_2, \ldots\ldots p_N\}$. And each delivers I_k bits of information, then the total amount of information for the set is $I_1(p) = \sum_{k=1}^{N} p_k I_k$

Appling the definition to the I(p) we got $I(p) = g^{-1}(\sum_{k=1}^{N} p_k g(I_k))$

When the postulate of additively for independent events is applied we get just two possible g(x):

g(x)=cx

$g(x)=c^{-2(1-\alpha)x}$

The first form gives Shannon information and the second gives $I_\alpha(p) = \frac{1}{1-\alpha} \log \sum_{k=1}^{N} P_k^{\alpha}$

In continuous distribution

$$I_\alpha(p) = \frac{1}{1-\alpha} \log \int_0^\infty g^p(x) dx, \alpha > 0, \alpha \neq 0$$

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Hence,

$$I_{\alpha}(p)=\frac{1}{1-\alpha}\log\int_{a}^{b}\left(\frac{\lambda}{(b-x)^{2}}\right)^{p}e^{p\lambda\left[\frac{a-x}{b-x}\right]}dx$$

The following Table below gives the values of Renyi Entropy of GU {EXP} distribution for different values of the parameters.

TABLE.3 Renyi Entropy

Parameters (a, b, λ)	p=2	p=3	p=4
a=0,b=3,λ=2	2.165886	1.493008	1.228449
a=0,b=3,λ=3	1.88563	1.147635	0.861639
a=0,b=3,λ=4	1.652627	0.8862	0.591115
a=0,b=3,λ=5	1.459951	0.677872	0.377719
a=0,b=3,λ=6	1.297103	0.505087	0.201692
a=0,b=3,λ=2.5	2.020267	1.307297	1.029283
a=0,b=3,λ=1	2.491538	1.941983	1.74671
a=0,b=3,λ=1.5	2.317845	1.707573	1.469053

6-The mean Residual life:

The mean Residual life (MRL) or the life expectancy at age t is the expected additional life length for a unit, which is alive at age t. the MRL is given by:

$$m(t)=\left(\frac{1}{1-F(x)}\int_{t}^{\infty}xg(x)dx\right)-t$$

The following Table below gives the values of MRL of GU {EXP} distribution for a fixed value of t.

TABLE.4 Mean residual life

Parameters (a, b, λ)	t=0.5	t=0.7
a=0,b=3,λ=2	0.503912	0.535269
a=0,b=3,λ=3	0.315346	0.407501
a=0,b=3,λ=4	0.209262	0.329167
a=0,b=3,λ=5	0.173057	0.300427
a=0,b=3,λ=6	0.144225	0.276369
a=0,b=3,λ=2.5	0.120977	0.255927
a=0,b=3,λ=1	0.886864	0.773294

7-Methods of Estimation

In this section we estimate the parameters of the GU{EXP} distribution by 4 different methods using complete sample technique. These methods are: maximum likelihood (MLE), Least-squares (LS), weighted least squares and percentile based estimation. The performance of all methods is studied by the R software.

7.1-Maximum likelihood estimation (MLE):

The MLE method is a general method and its estimators have some optimum properties such as consistency, asymptotic efficiency and invariance property.

Let $x_1, x_2, \ldots, x_n$ be a random sample from GU {EXP} population with PDF g(x) given in (5) with unknown parameters a, b, λ and α and the log-likelihood function is l(□, □, λ, α) then the MLE estimates of a, b, λ and α are the simultaneously solution of the following equations:

$$\frac{\partial \ell(a,b,\lambda,\alpha)}{\partial a}=0, \quad \frac{\partial \ell(a,b,\lambda,\alpha)}{\partial b}=0, \quad \frac{\partial \ell(a,b,\lambda,\alpha)}{\partial \lambda}=0, \quad \frac{\partial \ell(a,b,\lambda,\alpha)}{\partial \alpha}=0$$

Which gives the MLE estimates $(\hat{a},\hat{b},\hat{\lambda},\hat{\alpha})$. These equations are solved numerically using R software.

7.2-Method of Ordinary Least Squares (L):

The best estimates according to LS method are those which minimize the following quantity:

$$Q_1=\sum_{i=1}^{n}(G(x_{(i)}-\frac{i}{n+1}))^2$$

With respect to a, b, λ and α.

Where $x_{(i)}$ is the I th orders statistic of GU {EXP}

7.3-Method of Weighted Least Squares (WLS):

The WLS estimators of a, b, λ and α of GU {EXP} distribution can be obtained by minimizing the quantity:

$$Q_2 = \sum_{i=1}^{n} \frac{(n+1)^2 (n+2)}{n-i+1} (G(x_{(i)} - \frac{i}{n+1}))^2$$

With respect to a, b, λ and α.

7.4-Method of Percentile Estimation (PCE):

This method introduced by kao [17-a, 17-b] the PCEs estimators of a, b, λ and α of GU {EXP} distribution can be obtained by minimizing the quantity:

$$Q = \sum_{i=1}^{n} [x_{(i)} - a(\frac{1-(1-\sigma)(i/n+1)}{1-(i/n+1)})^{\frac{1}{\sigma^2}}]$$

Where $x_{(i)}$ is the I th orders statistic of GU {EXP}

7.5-Simulation Study and Data Analysis

The aim of this section is to compare the performance of the methods of estimation, namely: MLE, MPS, LS, WLS, and PE for the GU-Exp 3th distribution which discussed in the previous section. A Monte Carlo study is employed to check the behavior of the proposed methods of estimation. Also, a real data set is analyzed for illustrative purpose. R-statistical programming language will be used for calculation.

7.5.1-Simulation Study

A simulation study is employed to compare the performance of proposed methods of estimation using Monte Carlo. The Monte Carlo process is carried by generating 1000 random data from the GU-Exponential 3th distribution with the following assumptions:

1. Sample sizes are $n = 25, 50, 75, 100$.
2. Assume the following selected cases of parameters a and b of the GU-Exponential 3th distribution:
   a. $\alpha = 1,\ b = 3,\ \lambda = 3$
   b. $a = 0.50,\ b = 1.5,\ \lambda = 2$
   c. $a = 0.50,\ b = 2.5,\ \lambda = 1.5$

Based on the generated data and applying different methods of estimation. All the means square error (MSE) and relative biases (BIAS) are reported from Table (5) to Table

(8) for five different methods of estimation.

Table (5): The MSE and BIAS for different estimates of the GU-Exp 3th distribution with different values for parameters a, b, $and\ \lambda$ at sample size $n = 100$.

Parameters	MLE		MPS		LSE		WLSE		PE	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
case |										
a = 1	4.60710	2.11921	6.30214	2.15150	0.00041	0.00547	0.00010	0.00239	0.00055	0.00069
b = 3	0.13186	0.04061	6.00974	0.09818	44.15742	0.36360	29.01740	0.29047	25.01956	0.30793
λ = 3	0.30420	0.15694	8.14264	0.22057	37.88941	0.26585	34.46240	0.27582	37.26607	0.36043
case ||										
a = 0.5	2.46180	2.91881	1.96791	2.55996	0.00021	0.00731	5.20203	3.366616e	0.00021	0.00132
b =1.5	0.83046	0.38137	1.58173	0.25380	8.70356	0.41719	7.24085	0.32811	7.43282	0.32912
λ = 2	0.25673	0.09824	7.19136	0.67478	20.63015	0.46046	13.97240	0.35656	14.22655	0.37165
case |||										
a = 0.5	7.75773	5.15654	6.10884	4.54791	0.00154	0.02515	0.00038	0.01133	0.00131	0.00959
b =2.5	50.71899	1.18470	8.50305	0.24967	34.95956	0.62878	19.21114	0.42010	22.20501	0.44404
λ = 1.5	48.99015	2.84107	16.81716	1.21527	11.21368	0.59531	7.86033	0.43792	8.52236	0.46508

Table (6): The MSE and BIAS for different estimates of the GU-Exp 3th distribution with different values for parameters a, b $and\ \lambda$ at sample size $n = 75$.

Parameters	MLE		MPS		LSE		WLSE		PE	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
case I										
a = 1	4.73135	2.14184	4.87755	2.06641	0.00062	0.00626	0.00018	0.00239	0.00079	0.00128
b = 3	0.16533	0.99645	8.79210	0.08715	43.27392	0.30539	23.30787	0.17252	39.78318	0.29774
λ = 3	0.33989	0.15932	2.69164	0.13653	27.06938	0.13514	18.43761	0.08228	44.10905	0.26206
case II										
a = 0.5	2.13598	2.77182	2.65124	2.68644	0.00032	0.01028	9.29214	4.89891	0.00031	0.00172
b =1.5	0.61970	0.32668	2.71303	0.32257	11.09791	0.49187	10.41068	0.45782	7.00569	0.37981
λ = 2	0.20581	0.06764	10.10326	0.73387	17.93902	0.41675	18.80829	0.43853	18.60910	0.48848
case III										
a = 0.5	7.36311	4.97699	5.78569	4.48066	0.00212	0.02468	0.00062	0.01126	0.00166	0.00571
b =2.5	40.83684	0.98446	5.21153	0.19860	21.96843	0.37782	415.87186	0.02728	20.49737	0.39773
λ = 1.5	41.49535	2.53450	18.03436	1.31718	8.18946	0.35379	95.15548	0.03809	8.72367	0.43024

Table (7): The MSE and BIAS for different estimates of the GU-Exp 3th distribution with different values for parameters a, b $and\ \lambda$ at sample size $n = 50$.

Parameters	MLE		MPS		LSE		WLSE		PE	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
case I										
a = 1	4.87594	2.16705	5.88079	2.11594	0.00106	0.00805	0.00036	0.00313	0.00121	0.00241
b = 3	0.20784	0.05660	17.28998	0.17039	45.68617	0.33493	35.02276	0.23181	99568.20	3.88068
λ = 3	0.40182	0.16260	6.85878	0.17875	52.99073	0.29888	29.47157	0.13528	292946.54	6.91427
case II										
a = 0.5	2.15672	2.75323	2.54858	2.67085	0.00080	0.01460	0.00021	0.00688	0.00053	0.00098
b =1.5	1.59074	0.34088	5.04008	0.39624	11.75840	0.42956	6.86662	0.24456	8.13394	0.35381
λ = 2	0.31210	0.06396	18.34690	0.94876	24.74325	0.40876	12.81949	0.23278	14.38399	0.35025
case III										
a = 0.5	6.41647	4.69550	5.93402	4.46536	0.00680	0.04762	0.00152	0.02030	0.00322	0.00854
b =2.5	30.09203	0.67414	7.54109	0.22581	40.06765	0.58356	18.05652	0.32081	31.71282	0.47370
λ = 1.5	29.50171	2.05117	14.88996	1.22872	10.93833	0.46752	5.76064	0.30075	9.29851	0.45531

Table (8): The MSE and BIAS for different estimates of the GU-Exp 3th distribution with different values for parameters a, b $and\ \lambda$ at sample size $n = 25$.

Parameters	MLE		MPS		LSE		WLSE		PE	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
case I										
a = 1	5.08004	2.19986	5.58572	2.06250	0.00414	0.01997	0.00146	0.01009	0.00278	0.00210
b = 3	0.28676	0.06889	15.48126	0.16171	44.71596	0.31557	39.74692	0.23103	43.42958	0.21009
λ = 3	0.50863	0.16297	6.47334	0.26063	48.33705	0.30198	23.25090	0.08091	28.97423	0.09624
case II										
a = 0.5	1.84565	2.60362	2.25186	2.58324	0.00264	0.03416	0.00084	0.01685	0.00109	0.00487
b =1.5	0.95651	0.26460	6.50584	0.41185	12.77910	0.47459	9.84254	0.40454	8.38831	0.28773
λ = 2	3.51458	2.61914	13.78957	0.96000	13.35670	0.23593	19.45239	0.39548	18.97482	0.34716
case III										
a = 0.5	5.86164	4.53562	5.99729	4.43489	0.04143	0.12987	0.01767	0.06664	0.00739	0.01932
b =2.5	15.85724	0.41078	13.66782	0.30370	30.50163	0.45218	28.90815	0.41537	22.37131	0.33386
λ = 1.5	17.37909	1.65505	14.83158	1.39814	10.38693	0.32784	6.84585	0.26421	8.09956	0.33749

7.5.2 -Data Analysis

From the above tabulated results, one can indicate that:

The MSEs decrease as sample size increases.

Comparing the different methods of estimation, the results show that the WLSE produces the best results for estimating the parameter a and the MLE produces the best results for parameters (b, λ).

The ordering performance of the estimators in terms of LSs (from best to worst) for a WLSE and PE. For b is MLE and MPS. . While the ordering performance for λ is MLE, WLSE and LSE

8-Censored samples and types of censoring

In life testing experiments, items drawn from a population are put to test and their times to failure are noted. The test is usually terminated after a fixed time or after a fixed number of failures is observed, giving a censored sample.

The situation in which we observe the lifetimes of all the items is called uncensored data, but such data will rarely arise in reliability testing even under controlled conditions. It is often necessary to terminate the test before all failure items have been observed because of limited budget or time.

An observation is said to be right censored at L in the exact value of the observation is not known but it is known only that it is greater than or equal to L . similarly, an observation is said to be left censored at M if it is known only that the observation is less than or equal to M. right censoring is very common in lifetime data, but left censoring is fairly rare.

Some methods by which the data may be censored are:

Type I censored data

A life test for a fixed number n of items is terminated when all the items have failed or at a reassigned time t_0, whichever is the sooner. In this case the number of failures r is a random variable and t_0 is fixed.

Type I I censored data

A life test on n items is terminated after a predetermined number of failures r have occurred (r < n). Here the observations will present themselves as an ordered sequence, the shortest life time being observed first and the r-th shortest being observed last. In this case r and n are fixed beforehand and $X_{(r)}$, the time of the r-th failure, is a random variable.

Table. 9. MLE Estimation under type-I censoring scheme. n=100, 75

	n= 100			n= 75		
	a = 1	b = 3	λ = 3	a = 1	b = 3	λ = 3
Time of censoring (%) 30						
Bias	3.585117	0.590925	0.26794	3.560683	0.590093	0.249257
MSE	12.92868	3.143208	2.289198	12.75981	3.134569	2.197102
CP	100	95.3	100	100	93	100
Time of censoring (%) 60						
Bias	2.632122	0.507661	0.153504	2.64704	0.5073	0.108132
MSE	7.558645	2.319555	9.529201	7.66516	2.318513	6.219002
CP	98.6974	99.3	97.2	98.1982	99.8	96.8969
Time of censoring (%) 90						
Bias	2.161102	0.44225	0.077692	2.186031	0.450219	0.123005
MSE	6.260895	1.825179	6.306724	6.585706	1.874721	10.52389
CP	96.38554	99.4	97.58794	96.87815	99.49799	97.58551
Time of censoring (%) 99.99						
Bias	2.36431	0.448842	0.021109	2.381501	0.458286	0.0385
MSE	9.021046	1.853931	4.551557	9.003122	1.929847	6.511831
CP	95.7	97.2	98.6	95.4	96.8	98.0981

Table. 10. MLE Estimation under type-I censoring scheme. n=50, 25

	n= 50			n= 25		
	a = 1	b = 3	λ = 3	a = 1	b = 3	λ = 3
Time of censoring (%) 30						
Bias	3.586831	0.587621	0.287591	3.578505	0.587055	0.25223
MSE	13.22542	3.108982	2.140254	12.93515	3.103069	1.91954
CP	99.88263	85.3	100	97.14286	84.4	100
Time of censoring (%) 60						
Bias	2.603045	0.508069	0.193213	2.648492	0.510249	0.088862
MSE	7.713741	2.323815	11.53295	8.577409	2.349419	9.065901
CP	97.99398	99.6	95.996	97.8852	99.6994	96.69007
Time of censoring (%) 90						
Bias	2.211218	0.457252	0.088706	2.415364	0.465151	0.074149
MSE	6.292168	1.929401	4.816311	7.908364	2.728608	6.438343
CP	96.57603	99.4985	97.19157	95.57789	99.8998	97.99599
Time of censoring (%) 99.99						
Bias	2.503567	0.468716	0.01066	2.222892	0.483331	0.045082
MSE	9.950355	2.020889	3.768408	7.81253	2.158162	7.220962
CP	95.3954	97.2	97.8	95.2953	96	98.2983

Table. 11 . MLE Estimation under type-II censoring scheme. n=100, 75

	n= 100			n= 75		
	a = 1	b = 3	λ = 3	a = 1	b = 3	λ = 3
Time of censoring (%) 30						
Bias	3.277581	0.592117	0.052926	3.289193	0.583604	0.006167
MSE	10.99983	3.156705	4.532852	11.63553	3.548914	7.085122
CP	99.6997	98	97.89579	99.29789	99.9	97.7
Time of censoring (%) 60						
Bias	2.917295	0.483288	0.091276	2.86731	0.483258	0.072923
MSE	10.40286	5.1149	2.657811	9.86771	3.757724	4.639296
CP	97.04985	99.6994	98.79154	97.54098	99.39759	99.08815
Time of censoring (%) 90						
Bias	2.151505	0.441781	0.008632	2.048882	0.449128	0.052262
MSE	7.168615	1.792126	3.672995	6.373867	1.856627	3.752439
CP	96.4539	99.69849	97.89368	96.74797	99.59718	97.09419
Time of censoring (%) 99.99						
Bias	2.137646	0.446501	0.158651	2.172385	0.449195	0.084008
MSE	6.481329	1.832525	8.050975	6.758843	2.566496	4.458427
CP	95.4	97.4	97.3974	95.19038	99.8999	97.48996

Table. 12 . MLE Estimation under type-II censoring scheme. n=50, 25

	n= 50			n= 25		
	a = 1	b = 3	λ = 3	a = 1	b = 3	λ = 3
Time of censoring (%) 30						
Bias	3.473863	0.576603	0.00945	3.931077	0.552454	0.030614
MSE	14.42686	4.360896	4.999303	21.26886	4.81286	5.018848
CP	96.58635	99.8	98.3	94.12955	99.4	98.4
Time of censoring (%) 60						
Bias	2.921396	0.492549	0.06803	2.835477	0.464161	0.058817
MSE	11.04457	3.285291	4.812275	10.63685	8.378498	7.477346
CP	97.71784	99.7994	98.59155	98.17008	99.6997	98.9899
Time of censoring (%) 90						
Bias	2.18772	0.456261	0.05707	2.156509	0.460628	0.076823
MSE	6.928182	1.917124	7.452378	6.683976	3.236537	6.293035
CP	95.32995	99.69758	98.49246	96.32653	99.59555	97.69308
Time of censoring (%) 99.99						
Bias	2.270886	0.464862	0.186992	2.321561	0.478321	0.127239
MSE	7.429919	1.99411	14.13987	7.834517	2.114446	10.45399
CP	94.19419	97.1	98	93.96378	96.38191	98.29659

9- Real data

We use the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Ogunde [2]. The starting point of the iterative processes for the guinea pigs data set is (1:0; 0:009; 10:0; 0:1; 0:1). Survival Times (in days) of Guinea Pigs Infected with Virulent Tubercle Bacilli. The data is give as: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08,1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24,1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63,1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02,2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53,2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32,4.58, 5.55. Table 15. gives the MLEs and the selection criteria statistics for the pig data. Figure 3 gives the graph of the Total Time on Test plot and the graph of the kernel density of the pig data which clearly shows that the data exhibits an increasing failure rate and positively skewed (unimodal) also Figure 3 gives the fitted densities of the pig data.

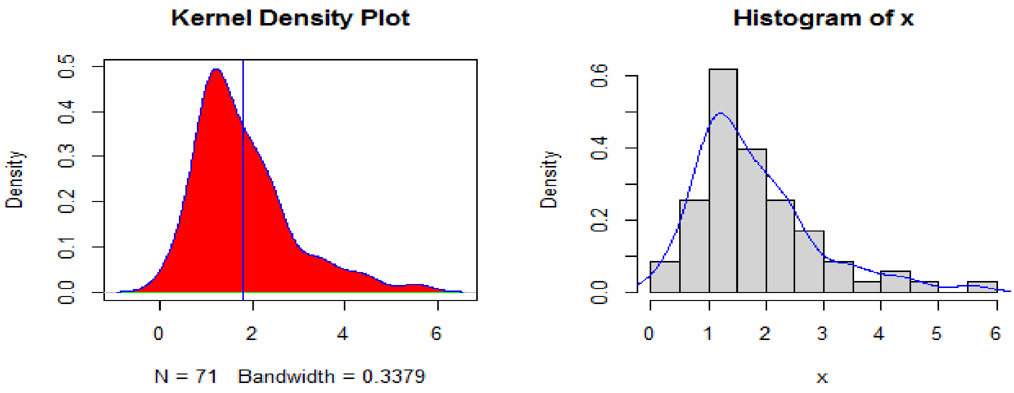

Figure -3. The kernel density and the fitted densities of the pig data.

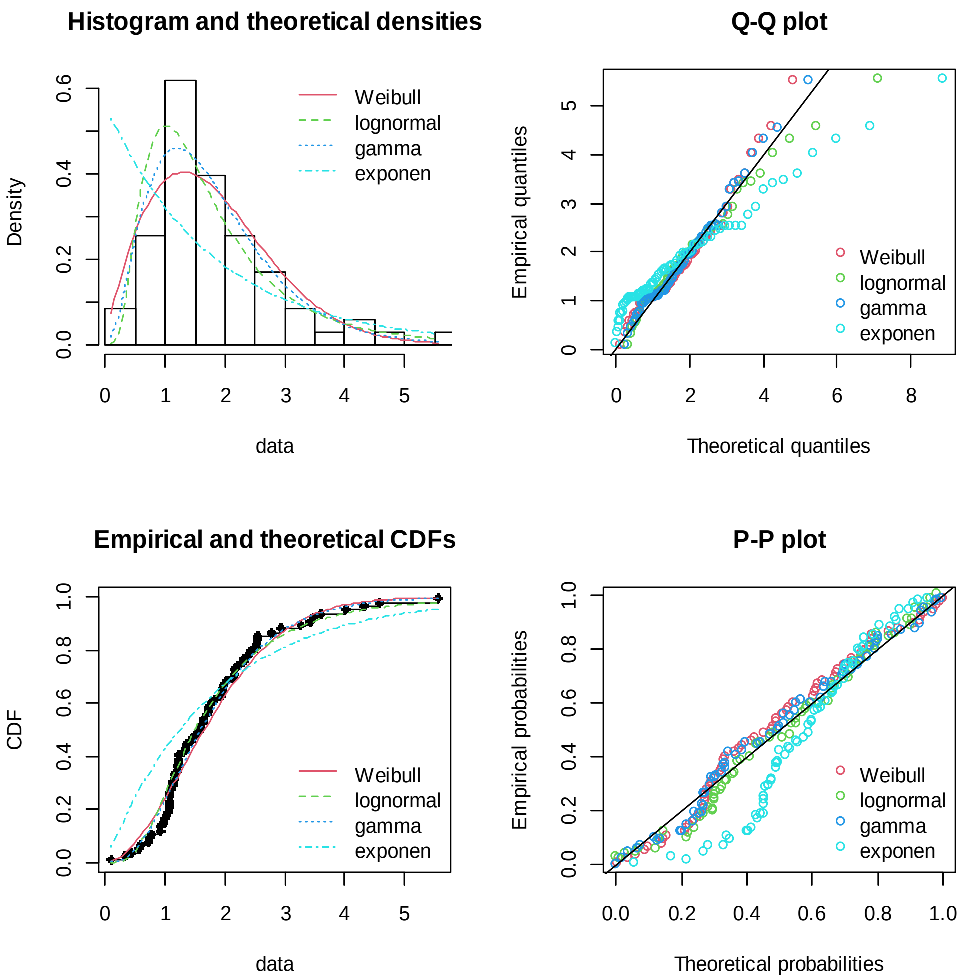
Histogram and theoretical densities
Density
data
Weibull
lognormal
gamma
exponen
Q-Q plot
Empirical quantiles
Theoretical quantiles
Empirical and theoretical CDFs
CDF
P-P plot
Empirical probabilities
Theoretical probabilities

Table.15 Parameter estimate, standard error and Selection criteria statistics

distribution	Estimates			Log-Likelihood	AIC	BIC
GU{EXP}	2.7792 (8.7074)	2.7800 (5.303)	7.3544 (3.212)	-82.55	137.1029	141.6282
Gompertz	0.44998 (0.0903)	0.29019 (0.058)		-101.7625	207.5249	NA
gamma	3.081174 (0.4917)	1.72470 (0.298)		-93.67446	191.3489	195.8743
lognormal	0.409366 (0.0750)	0.63221 (0.053)		-97.26063	198.5213	203.0466
weibull	1.836293 (0.16125)	2.01660 (0.1377)		-94.98842	193.9768	198.5022
exponential	0.559716 (0.06642)	- -		-112.2031	226.4062	228.6689

Referance

1- Alfred Renyi, On Measures of entropy and information, 1961
2- A. A. Ogunde, G. A. Olalude, D. O. Omosigho, The Gompertz Gumbel II Distribution: Properties and Applications, Academic Journal of Applied Mathematical Sciences, Vol. 7, Issue. 1, pp: 1-15, 2021
3-A. Alzaghal, F. Famoye, C.Lee: Exponentiated T-X family of distributions with some applications. Int. J. Stat. Probab. 2 (2013), 31–49.
4- A. K. Srivastava, Statistical Analysis and Model Validation of Gompertz Model on different Real Data Sets for Reliability Modelling, Journal of Advanced Computing and Communication Technologies, Volume No.1 Issue No.2., December 2013, pp. 371–381.
5- A.El-Gohary , A. Alshamrani, A. Al-Otaibi, The generalized Gompertz distribution, Applied Mathematical Modelling 37 (2013),pp. 13–24
6-A.H.El-Bassiouny, M.EL-Damcese, A.f. Mustafa and M.S.Eliwa, Exponentiated Generalized Weibull-Gompertz Distribution with Application in Survival Analysis, J. Stat. Appl. Pro. 6, No. 1, 7-16 (2017)
7- A.Jafari, S.Tahmasebi, M.Alizadeh, The Beta-Gompertz Distribution, Revista Colombiana de Estadística, Junio 2014, volume 37, no.1, pp. 141-158
8- B. Gompertz, on the nature of the function expressive of the law of human mortality and on The new mode of determining the value of life contingencies, Philosophical Transactions of Royal Society A115 (1825) 513–580.
9- E. A. Ahmed, Estimation of some life time parameters of generalized Gompertz distribution under progressively type-II censored data, Applied Mathematical Modeling 39 (2015), 5567–5578
10- F. J. Torres, Estimation of parameters of the shifted Gompertz Distribution using least squares, maximum likelihood and Moments methods, Journal of Computational and Applied Mathematics 255 (2014) 867–877
11- F.Chipepaa,B.Oluyedea, The Marshall-Olkin-Gompertz-G family of distributions: properties and applications, J.Nonlinear Sci. Appl., 14 (2021), 250–267

12 - H. H. Abu-Zinadah and A. S. Aloufi, Some Characterizations of the Exponentiated Gompertz Distribution, International Mathematical Forum, Vol. 9, 2014, no. 30, 1427 – 1439
13- H. S. Bakouch, A. M. T. AbdEl-Bar, A NEW WEIGHTED GOMPERTZ DISTRIBUTION WITH APPLICATIONS TO RELIABILITY DATA, APPLICATIONS OF MATHEMATICS, 62(2017), No.3,269–296
14- I. B. Abdul-Moniem, M. Seham2, Transmuted Gompertz Distribution, Computational and Applied Mathematics, 2015; 1(3): 88-96
15- J. C. Ahuja and S. W. Nash, The Generalized Gompertz-Verhulst Family of Distributions, The Indian Journal of Statistics, Series A, Vol. 29, No. 2 (Jun،.1967), pp. 141-156
16- J.W.Wu ,W.L.Hung , C.H.Tsai, Estimation of parameters of the Gompertz distribution using the least squares method, Applied Mathematics and Computation 158 (2004) pp.133–147
17- J.W.Wu, W.L.Hung, and C.H.Tsai, Estimation of the parameters of the Gompertz distribution under the first failure-censored sampling plan, Statistics 37 (6) (2003), pp.517–525.
18- Khaleel,M.A.,Oguntunde,P.E.,Ahmed,M.T.,Ibrahim,N.A.,andLoh, Y.F., The Gompertz Flexible Weibull Distribution and Its Applications, Malaysian Journal of Mathematical Sciences 14(1) (2020): 169190
19- M. Alizadeh, G. M. Cordeiro, L. G. B. Pinho & I. Ghosh: The Gompertz-G family of distributions, Journal of Statistical Theory and Practice, (2017), DOI: 10.1080/15598608.2016.1267668
20- M. A. Khaleel, N. H. Al-Noor, M. Kh. Abdal-Hameed, Marshall Olkin exponential Gompertz distribution: Properties and applications, Periodicals of Engineering and Natural Sciences, Vol. 8, No. 1, March 2020, pp.298-312.
21- M. Alizadeh, G. M. Cordeiro, L. G. B. Pinho & I.Ghosh ,The Gompertz-G family of distributions, Journal of Statistical Theory and Practice، 11:1, (2017) 179-207, DOI: 10.1080/15598608.2016.1267668
22- M. E. Ghitany, F. Alqallaf& N. Balakrishnanb, On the likelihood estimation of the parameters of Gompertz distribution based on complete and progressively Type-II censored samples, Journal of Statistical Computation and Simulation, 84:8, (2014), 1803-1812.

23- M.L.Garg, B.R.Rao, C.K.Redmond, Maximum likelihood estimation of the parameters of The Gompertz survival function, Journal of the Royal Statistical Society C19 (1970) 152–159.

24 - N. H. Gordon: Maximum likelihood estimation for mixtures of two gompertz distributions when censoring occurs, Communications in Statistics - Simulation and Computation, 19:2(1990), 733-747

25- P. Oguntunde, M. Khaleel, H. Okagbue, A. Opanuga and F. Owolabi, The gompertz inverse exponential (GoIE) distribution with applications, Cogent Mathematics & Statistics, Volume 5, 2018 - Issue 1,

26- P.H.Franses, Fitting a Gompertz curve, Journal of the Operational Research Society 45 (1994) 109–113.

27- R. Roozegar, S. Tahmasebi & A. Jafari, The McDonald Gompertz Distribution: Properties and Applications, Communications in Statistics - Simulation and Computation, Volume 46, 2017 - Issue 5, Pages 3341-3355.

28- S.J.WU, C.TAO, T.R. TSAI, Point and interval estimations for the Gompertz distribution under progressive type-II censoring, METRON- International Journal of Statistics, 2003, vol. LXI, n.3, pp.403-418

29- T. G. Ieren, F. M. Kromtit, B. U. Agbor, I. B. Eraikhuemen and P. O. Koleoso4, A Power Gompertz Distribution: Model, Properties and Application to Bladder Cancer Data, Asian Research Journal of Mathematics, 15(2): 1-14, 2019

30-Y. Wang and W. Gui, Estimation and Prediction for Gompertz Distribution under General Progressive Censoring, Symmetry (2021), 13, 858.

31-Z.Chen, Parameter estimation of the Gompertz population, Biometrical Journal 39 (1997) 117–124

DOI: 10.1080/00949655.2013.766738