A New Family of Discrete Alpha Power Distributions

A. A. EL-Helbawy[1], M. A. Hegazy[2] and G. R. AL-Dayian[1]

[1]*Department of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt*

[2]*Department of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Tafahna Al-Ashraf, Egypt*

Abstract

In this paper, a new family of discrete alpha power distributions is introduced. Some properties including quantiles, mean residual life, mean time to failure, Rényi entropy, moments and order statistics are obtained. Discrete alpha power Weibull distribution, as a member from this family, is studied in detail. Discrete two-parameter Weibull distribution, discrete alpha power one parameter Weibull distribution, discrete alpha power exponential distribution, discrete one parameter Weibull distribution, discrete Rayleigh distribution, discrete exponential distribution, discrete alpha power Rayleigh distribution are sub models of discrete alpha power Weibull distribution. A simulation study is conducted to investigate the precision of the theoretical results based on simulated and real data through some measurements of accuracy. Three real data sets are analyzed to illustrate the suitability and applicability of the proposed model.

Keywords: *Alpha power transformation; Discrete distributions; Discrete alpha power family of distributions; Weibull distribution; Maximum likelihood estimation*.

1. Introduction

Generalization for classical distributions has received much attention in recent years by many authors to let the extended distributions more flexible for modeling real data. In practice the motivations for obtaining generalized family are: (a) to make the kurtosis more flexible as compared to the baseline model, (b) to produce skewness for symmetrical distributions, (c) to construct heavy-tailed distributions that are not longer-tailed for modeling real data, (d) to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped, (e) to provide consistently better fits than other generated models under the same underlying distribution. [See Eliwa *et al.* (2020)].

In the literature, several methods of generating new family of statistical distributions were presented; for example, Marshall and Olkin (1997), Eugene *et al.* (2002), Cordeiro and Castro (2011), Alzaatreh *et al.* (2013), Lee *et al.* (2013) and Jones (2015).

Mahdavi and Kundu (2017) presented a method to add an extra parameter to a family of distributions, such an addition of parameters makes the resulting distribution richer and more flexible for modeling data. The suggested method is called *alpha power transformation* (APT) and it is useful to incorporate skewness to a family of distributions. The APT method was applied to many distributions by many researchers, such as Nassar *et al.* (2017), Dey *et al.* (2017), Nadarajah and Okorie (2018), Mead *et al.* (2019) and Nassar *et al.* (2020).

Let $F(x)$ is the *cumulative distribution function* (cdf) and the APT of $F(x)$ for $x \in \mathbb{R}$ is

$$G_{APT}(x;\alpha)=\begin{cases}\dfrac{\alpha^{F(x)}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\ F(x), & \alpha=1,\end{cases} \tag{1}$$

and the corresponding *probability density function* (pdf) is

$$g_{APT}(x;\alpha)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}f(x)\alpha^{F(x)}, & \alpha>0,\ \alpha\neq 1,\\ f(x), & \alpha=1,\end{cases} \tag{2}$$

where α is the shape parameter.

The *survival function* (sf); $S_{APT}(x;\alpha)$, is given by

$$S_{APT}(x;\alpha)=\begin{cases}\frac{\alpha}{\alpha-1}\left(1-\alpha^{F(x)-1}\right), & \alpha>0,\ \alpha\neq 1,\\ 1-F(x), & \alpha=1,\end{cases} \tag{3}$$

In reliability lifetime modeling, it is common to deal with failure data as continuous but practically; failures can happen and are observed in a discrete procedure. Some well-known discrete distributions have limited applicability to model discrete failure times.

Therefore, it is realistic and suitable to generate discrete lifetime distribution from the base continuous distribution keeping one or more important characters of the continuous distribution. [For more details see, Lai (2013) and Chakraborty and Chakravorty (2016)]. Although there are several methods where discrete analogue random variable of a continuous random variable may be obtained, the general approach of discretization of some known continuous distributions have been attracting great concern for use as lifetime distributions [see, Nakagawa and Osaki (1975), Khan *et al.* (1989), Bracquemond and Gaudoin (2003), Inusah and Kozubowski (2006), Krishna and Pundir (2009), Jazi *et al.* (2010), Gomez-Deniz and Calderin-Ojeda (2011) and Nekoukhou *et al*. (2012) and Chakraborty (2015)].

The rest of this paper is organized as follows: in Section 2, a *discrete alpha power* (DAP) family of distributions based on the method of APT is introduced and some of its properties are studied. In Section 3, some members of the discrete family of distributions are presented. *Maximum likelihood* (ML) estimation for the parameters of the distribution is discussed in Section 4. In Section 5, real data set is analyzed to demonstrate how the results can be used in practice. Finally, concluding remarks are given in Section 6.

2. Discrete Alpha Power Family of Distributions: Construction and Properties

In this section, DAP family of distributions is constructed using the general approach of discretizing, where the advantage of this method is that the sf for the discrete distributions has the same functional form of the sf for the continuous distributions. Hence many reliability and properties keep on unchanged [see, Roy (2003, 2004)].

2.1 Discrete Alpha Power family of distributions

If the continuous random variable X has the sf, $S(x) = P(X \geq x)$ then the *probability mass function* (pmf) of the *discrete* $X(dX)$ is given by

$$P(dX = x) = P(x) = P[x \leq X \leq x + 1]$$
$$= S(x) - S(x+1), \qquad x = 0, 1, 2, \dots \; . \quad (4)$$

Considering X is a discrete random variable analogue to a continuous random variable. From (3) and (4), the pmf and cdf for DAP family are given, respectively, by

$$P_{DAP}(x) \equiv P_{DAP}(x;\alpha) = S_{DAP}(x) - S_{DAP}(x+1)$$
$$= (\alpha-1)^{-1}\left(\alpha^{F(x+1)} - \alpha^{F(x)}\right),\quad x = 0,1,2\ldots\ ;\ \alpha \neq 1, \qquad (5)$$

and

$$F_{DAP}(x;\alpha) = P_{DAP}(X \le x) = 1 - S_{DAP}(x) + P_{DAP}(X = x)$$
$$= \frac{\alpha^{F(x+1)}-1}{\alpha-1},\quad x = 0,1,2\ldots\ ;\ \alpha \neq 1\,. \qquad (6)$$

2.2 Some properties

1. Survival and hazard rate functions

The sf and the corresponding *hazard rate function* (hrf) are given below

$$S_{DAP}(x) = P_{DAP}(X \ge x) = 1 - F_{DAP}(x) + P_{DAP}(X = x)$$
$$= \frac{\alpha}{\alpha-1}\left(1-\alpha^{F(x)-1}\right),\qquad x = 0,1,2\ldots;\ \alpha \neq 1, \qquad (7)$$

and

$$h_{DAP}(x;\alpha) = \frac{P_{DAP}(x)}{S_{DAP}(x)} = \frac{\left(\alpha^{F(x+1)}-\alpha^{F(x)}\right)}{\alpha\left(1-\alpha^{F(x)-1}\right)},\quad x = 0,1,2\ldots\ ;\ \alpha \neq 1. \qquad (8)$$

2. Reversed and alternative hazard rate function

The *reversed hazard rate function* (rhrf) which is known by the dual of the hrf; describes the probability of an immediate past failure, given that the unit has already failed at time x. The rhrf is given by

$$rh_{DAP}(x;\alpha) = \frac{P_{DAP}(x)}{F_{DAP}(x)} = \frac{\left(\alpha^{F(x+1)}-\alpha^{F(x)}\right)}{\alpha^{F(x+1)}-1},\quad x = 0,1,2\ldots\ ;\ \alpha \neq 1\,. \qquad (9)$$

The discrete hrf has some notable problems. Therefore, Roy and Gupta (1992) provided an excellent alternative definition of a discrete hrf; *alternative hazard rate function* (ahrf); $ah(x)$. The hrf and ahrf have the same monotonic property, i.e., ahrf is increasing (decreasing) if and only if hrf is increasing (decreasing). [For more details see, Xie *et al.* (2002) and Lai (2013, 2014)].

The alternative hrf is

$$ah_{DAP}(x;\alpha) = ln\left[\frac{S_{DAP}(x)}{S_{DAP}(x+1)}\right]$$
$$= ln\left[\frac{\left(1-\alpha^{F(x)-1}\right)}{\left(1-\alpha^{F(x+1)-1}\right)}\right],\quad x = 0,1,2\ldots\ ;\ \alpha \neq 1. \qquad (10)$$

The relationship between $ah_{DAP}(x)$ and $h_{DAP}(x)$ can be expressed by

$$h_{DAP}(x;\alpha) = 1 - e^{-ah_{DAP}(x;\alpha)}$$
$$= 1 - e^{-[ln(1-\alpha^{F(x)-1})-ln(1-\alpha^{F(x+1)-1})]} . \quad (11)$$

3. Quantile function

The pth quantile function, say x_p, is

$$x_p = F^{-1}\left[\frac{log[(\alpha-1)p+1]}{\log(\alpha)}\right] - 1 \ , \ \alpha \neq 1, \quad (12)$$

where $p \in (0,1)$, $x_p > 0$ and F^{-1}represents the base line of quantile function.

Special quantiles may be obtained using (12). For example, if $p = 0.5$, the median of the DAP distribution is

$$\text{Median} = F^{-1}\left[\frac{log[0.5(\alpha-1)+1]}{\log(\alpha)}\right] - 1 \ .$$

4. Mean residual life

The m*ean residual life* (MRL) is the expected remaining life, $X - x_0$, given that the item has survived to time x_0 [see, Kemp (2004)]. It is denoted by $m(x_0)$ and is defined by

$$m(x_0) = \frac{\sum_{k=x_0+1}^{\infty} S_{DAP}(k)}{S_{DAP}(x_0)} = \frac{\sum_{k=x_0+1}^{\infty}(1-\alpha^{F(k)-1})}{(1-\alpha^{F(x_0)-1})} \ . \quad (13)$$

5. Mean time between failures and mean time to failure

Mean Time to Failure (MTTF) is the average time between non-repairable failures and is generally used for items that cannot be repaired, such a light bulb or a backup tape. The average time for a device or system is expected to function before it fails. It predicts the failure rate for products that cannot be repaired.

The MTTF is given as follows:

$$MTTF = \sum_{t=1}^{\infty} S_{DAP}(t) = \sum_{t=1}^{\infty} \frac{\alpha}{\alpha-1}\left(1-\alpha^{F(t)-1}\right) \ , \ t > 0; \ \alpha \neq 1. \quad (14)$$

The *Mean Time between Failure* (MTBF) is used with items that can be either repaired or replaced and is given bellow

$$MTBF = \frac{-t}{log[S_{DAP}(t)]} = \frac{-t}{log[(\frac{\alpha}{\alpha-1})(1-\alpha^{F(t)-1})]} \ , \ t > 0; \ \alpha \neq 1. \quad (15)$$

The *Availability* (Av) is considered as being the probability that the component is successful at time x, i.e.,

$$Av = \frac{MTTF}{MTBF} \quad . \tag{16}$$

It's important for organizations to be aware of the difference between the three previous concepts, so they don't waste time focusing on how long it takes to repair a system when the best option could be to replace it with a new one.

6. Rényi entropy

An entropy of a random variable X with the pdf $P(x)$, is a measure of variation of the uncertainty and it is denoted by $H_R(\rho)$. It has been applied in a wide variety of fields such as statistical thermodynamics, urban and regional planning, business, economics, finance, operations research, queueing theory, spectral analysis, image reconstruction, biology and manufacturing. It is defined by

$$H_R(\rho) = (1-\rho)^{-1} log\left\{\sum_{x=0}^{\infty} \left(P_{DAP}(x)\right)^{\rho}\right\}$$

$$= (1-\rho)^{-1} log\left\{\sum_{x=0}^{\infty} \left(\frac{\alpha^{F(x+1)} - \alpha^{F(x)}}{\alpha - 1}\right)^{\rho}\right\}, \alpha \neq 1 \ \rho > 0, \rho \neq 1 \ . \tag{17}$$

The Shannon entropy can be defined by $E\left[-log\left(P_{DAP}(X)\right)\right]$, and it can be calculated as a special case of the Rényi entropy when $\rho \to 1$.

7. Non-central and central moments

The non-central moments are obtained as follows:

$$\mu'_r = E(X^r) = \sum_{x=0}^{\infty} x^r \, P_{DAP}(x) \tag{18}$$

$$\mu'_r = \sum_{x=0}^{\infty} x^r \left[(\alpha-1)^{-1}\left(\alpha^{F(x+1)} - \alpha^{F(x)}\right)\right],$$

$$x = \ 0,1,2 \ldots ; \alpha \neq 1, r \ = 1, 2, \ldots \ . \tag{19}$$

In particular, the mean is given by

$$\mu'_1 = \mu = \sum_{x=0}^{\infty} x \left[(\alpha-1)^{-1}\left(\alpha^{F(x+1)} - \alpha^{F(x)}\right)\right], \qquad x = \ 0,1,2 \ldots ; \ \alpha \neq 1. \tag{20}$$

8. Order statistic

Let $F_{iDAP}(x)$; the cdf of the i^{th} order statistic for a random sample $X_1, X_2, \ldots, X_n$, is given by

$$F_{iDAP}(x) = \sum_{r=i}^{n} \binom{n}{r} \left[F_{DAP}(x)\right]^r \left[1 - F_{DAP}(x)\right]^{n-r}. \tag{21}$$

Using the binomial expansion for $[1 - F_{DAP}(x)]^{n-r}$ and substituting (6) in (21),
where

$$F_{iDAP}(x) = \sum_{r=i}^{n} \binom{n}{r} [F_{DAP}(x)]^r \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j [F_{DAP}(x)]^j$$

$$= \sum_{r=i}^{n} \binom{n}{r} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left[\frac{\alpha^{F(x+1)}-1}{\alpha-1}\right]^{r+j}, \; x = 0,1,2\dots; \; \alpha \neq 1. \quad (22)$$

Special cases

Case I: If $i = 1$ in (22) one can obtain the distribution function of the first order statistic, as given below

$$F_{1DAP}(x) = 1 - [1 - F_{DAP}(x)]^n$$

$$= 1 - \left[1 - \frac{\alpha^{F(x+1)}-1}{\alpha-1}\right]^n. \quad (23)$$

Case II: If $i = n$ in (22) the distribution function of the largest order statistic is as follows:

$$F_{nDAP}(x) = [F_{DAP}(x)]^n$$

$$= \left[\frac{\alpha^{F(x+1)}-1}{\alpha-1}\right]^n. \quad (24)$$

Suppose that $X_1, X_2, \dots, X_n$ is a random sample from the DAP distribution. Let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ denote the corresponding order statistics. [see Arnold *et al.* (2008)]. Then, the pmf of $X_{i:n}$, is defined by

$$P_{DAP}(X_{i:n} = x) = \frac{n!}{(i-1)!(n-i)!} \int_{F_{DAP}(x-1)}^{F_{DAP}(x)} v^{i-1}(1-v)^{n-i} dv.$$

(25)

Using the binomial expansion for $(1-v)^{n-i}$, then, the pmf in (25) is

$$P(X_{i:n} = x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left(\frac{1}{i+j}\right)$$

$$\times \left[\left(\frac{\alpha^{F(x+1)}-1}{\alpha-1}\right)^{i+j} - \left(\frac{\alpha^{F(x)}-1}{\alpha-1}\right)^{i+j}\right], \alpha \neq 1. \quad (26)$$

The pmf of the smallest order statistic is obtained by substituting $i = 1$ in (24) as given below

$$P_{DAP}(X_{1:n} = x) = [1 - F_{DAP}(x)]^n - [1 - F_{DAP}(x+1)]^n, \;\; \alpha \neq 1, \quad (27)$$

and, the pmf of the largest order statistic is obtained by substituting $i = n$ in (24) as follows:

$$P_{DAP}(X_{n:n} = x) = [F_{DAP}(x+1)]^n - [F_{DAP}(x)]^n, \;\; \alpha \neq 1. \quad (28)$$

Also, one can use (22) to obtain the pmf of the DAP distribution, [see Arnold *et al*. (2008)].

3. Some Members of Discrete Family of Distributions

In this section the DAP *transformation* (DAPT) method is applied to a specific class of distribution functions such as exponential, uniform and Weibull distributions.

3.1. Discrete alpha power exponential distribution

The cdf and sf of the exponential distribution with parameter λ are, respectively,

$$F(y;\lambda) = 1 - e^{-\lambda y}\ ,\ \ y > 0\ ;\ \lambda > 0\ ,$$

and

$$S(y;\lambda) = 1 - F(x;\lambda) = e^{-\lambda y},\ \ y > 0\ ;\ \lambda > 0\ .$$

Let $e^{-\lambda} = p_1$, $0 < p_1 < 1$.

Using (5)-(7), the pmf, cdf and sf of the two-parameter DAP exponential distribution are, respectively, given by

$$P_{DAPE}(y) \equiv P_{DAPE}(y;\alpha,\lambda) = \frac{\alpha}{\alpha-1}\left(\alpha^{-p_1^{(y+1)}} - \alpha^{-p_1^{y}}\right),\ \ y = \ 0,1,2 \dots\ ;\ \ \alpha \neq 1, \quad (29)$$

$$F_{DAPE}(y) = \frac{\alpha^{1-p_1^{(y+1)}} - 1}{\alpha - 1},\qquad y = \ 0,1,2 \dots\ ;\ \ \alpha \neq 1, \quad (30)$$

and

$$S_{DAPE}(y) = \left(\frac{\alpha}{\alpha-1}\right)\left(1 - \alpha^{-p_1^{y}}\right),\ \ y = \ 0,1,2 \dots\ ;\ \ \alpha \neq 1, \quad (31)$$

Note that: If $\alpha = 1$, the distribution with pmf (29) reduces to the discrete exponential distribution.

3.2 Discrete alpha power uniform distribution

Assuming that Y has uniform distribution with parameter a. Then the cdf and sf of Y are, respectively,

$$F(y;a) = \frac{y}{a}\ ,\qquad y < a\ ;a > 0\ ,$$

and

$$S(y;\lambda) = 1 - F(y;a) = 1 - \frac{y}{a},\qquad y < a\ ;\ \ a > 0\ .$$

Hence, the pmf, cdf and sf of the discrete alpha power uniform distribution using (5)-(7) are, respectively, given by

$P_{DAPU}(y) \equiv P_{DAPE}(y;\alpha,\lambda) = (\alpha-1)^{-1}\left(\alpha^{(y+1)/a} - \alpha^{y/a}\right), \quad y = 0,\dots,a\,; \quad \alpha \neq 1,$
(32)

$$F_{DAPU}(y) = \frac{\alpha^{(y+1)/a}-1}{\alpha-1}, \qquad y = 0,\dots,a\;; \quad \alpha \neq 1, \qquad (33)$$

and

$$S_{DAPU}(y) = \left(\frac{\alpha}{\alpha-1}\right)\left(1-\alpha^{y/a-1}\right), \quad y = 0,1,2\dots,a\;; \quad \alpha \neq 1, \qquad (34)$$

Note that: If $\alpha = 1$, the distribution with pmf in (32) reduces to a discrete uniform distribution.

4. Discrete Alpha Power Weibull Distribution

The cdf and sf of the Weibull distribution with scale parameter λ and shape parameter β are

$F(y;\ \lambda,\beta) = 1 - e^{-\lambda y^{\beta}},\ y > 0;\ \ \lambda,\beta > 0\,,$

and

$S(y;\lambda,\beta) = e^{-\lambda y^{\beta}}, \qquad y > 0;\ \ \lambda,\beta > 0\,.$

Let $e^{-\lambda} = p_2\ ,\qquad 0 < p_2 < 1\,.$

Then, the pmf and cdf of the discrete alpha power Weibull distribution using (5)-(7) are, respectively, given by

$P_{DAPW}(y) \equiv P_{DAPW}(y;\alpha,\lambda) = \left(\frac{\alpha}{\alpha-1}\right)\left(\alpha^{-p_2^{(y+1)^{\beta}}} - \alpha^{-p_2^{y^{\beta}}}\right), y = 0,1,2\dots; \beta > 0,\ \alpha \neq 1,$

(35)

and

$$F_{DAPW}(y) = \frac{\alpha^{1-p_2^{(y+1)^{\beta}}}-1}{\alpha-1}, \qquad y = 0,1,2\dots;\ \beta > 0,\ \ \alpha \neq 1. \qquad (36)$$

4.1 Survival, hazard, alternative and reversed hazard rate functions of discrete alpha power Weibull distribution

The sf, hrf, ahrf and rhrf are

$$S_{DAPW}(y) = \left(\frac{\alpha}{\alpha-1}\right)\left(1-\alpha^{-p_2^{y^{\beta}}}\right), \quad y = 0,1,2\dots;\ \beta > 0,\ \ \alpha \neq 1, \qquad (37)$$

$$h_{DAPW}(y) = \frac{\left(\alpha^{-p_2^{(y+1)^{\beta}}} - \alpha^{-p_2^{y^{\beta}}}\right)}{\left(1-\alpha^{--p_2^{y^{\beta}}}\right)}, \quad y = 0,1,2\dots\ ;\ \beta > 0,\ \ \alpha \neq 1, \ (38)$$

$$ah_{DAPW}(y) = ln\left[\frac{\left(1-\alpha^{-p_2^{y^\beta}}\right)}{\left(1-\alpha^{-p_2^{(y+1)^\beta}}\right)}\right] \quad , \; y = 0,1,2\ldots \; ; \beta > 0, \;\; \alpha \neq 1, \quad (39)$$

and

$$rh_{DAPW}(y) = \frac{\left(\alpha^{1-p_2^{(y+1)^\beta}} - \alpha^{1-p_2^{y^\beta}}\right)}{\alpha^{1-p_2^{(y+1)^\beta}} - 1} \; , \quad y = 0,1,2\ldots \; ; \beta > 0, \;\; \alpha \neq 1, \; (40)$$

4.2 Some sub-models of the discrete alpha power Weibull distribution

Some important special sub-models of the DAPW distribution are given in Table 1.

Table 1. Sub-models of the DAPW distribution

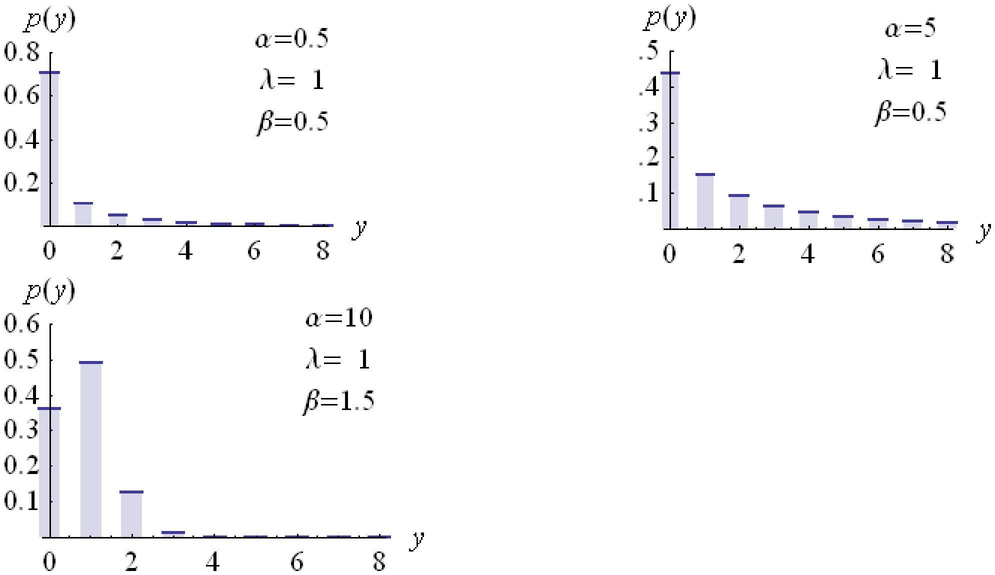
α	λ	β	Model
1	–	–	Discrete two parameter Weibull distribution
–	1	–	Discrete alpha power one parameter Weibull distribution
–	–	1	Discrete alpha power exponential distribution
1	1	–	Discrete one parameter Weibull distribution
1	–	2	Discrete Rayleigh distribution
1	–	1	Discrete exponential distribution
–	–	2	Discrete alpha power Rayleigh distribution

4.3 Graphical description

Figure 1 displays some plots of pmf of DAPW for selected parameter values. Plots of the hrf are given in Figure 2 and ahrf plots of the DAPW distribution for selected parameter values are presented in Figure 3.

Figure 1 shows that the pmf of DAPW distribution can be unimodel and right skewed according to the selected values of the parameters. For some values of parameters, the pmf is decreasing over $(0,\infty)$ and the mode is at zero. While for other values of the parameters, it indicates that the pmf is increasing on $(0, x_{mode})$ and reaches the maximum at x_{mode}, then decreases to the zero. Plots of pmf, hrf and ahrf show that the DAPW distribution exhibits a long right tail compared with other commonly used distributions. Thus, it will affect long term reliability predictions, producing optimistic predictions of rare events occurring in the right tail of the distribution compared with other distributions. Also, the DAPW distribution provides a good fit to several data in literature.

Figures 2 and 3 indicate that although the hrf and ahrf of DAPW distribution are decreasing, increasing and upside-down bathtub shapes depending on the value of the shape parameters, the hrf is less than 1.



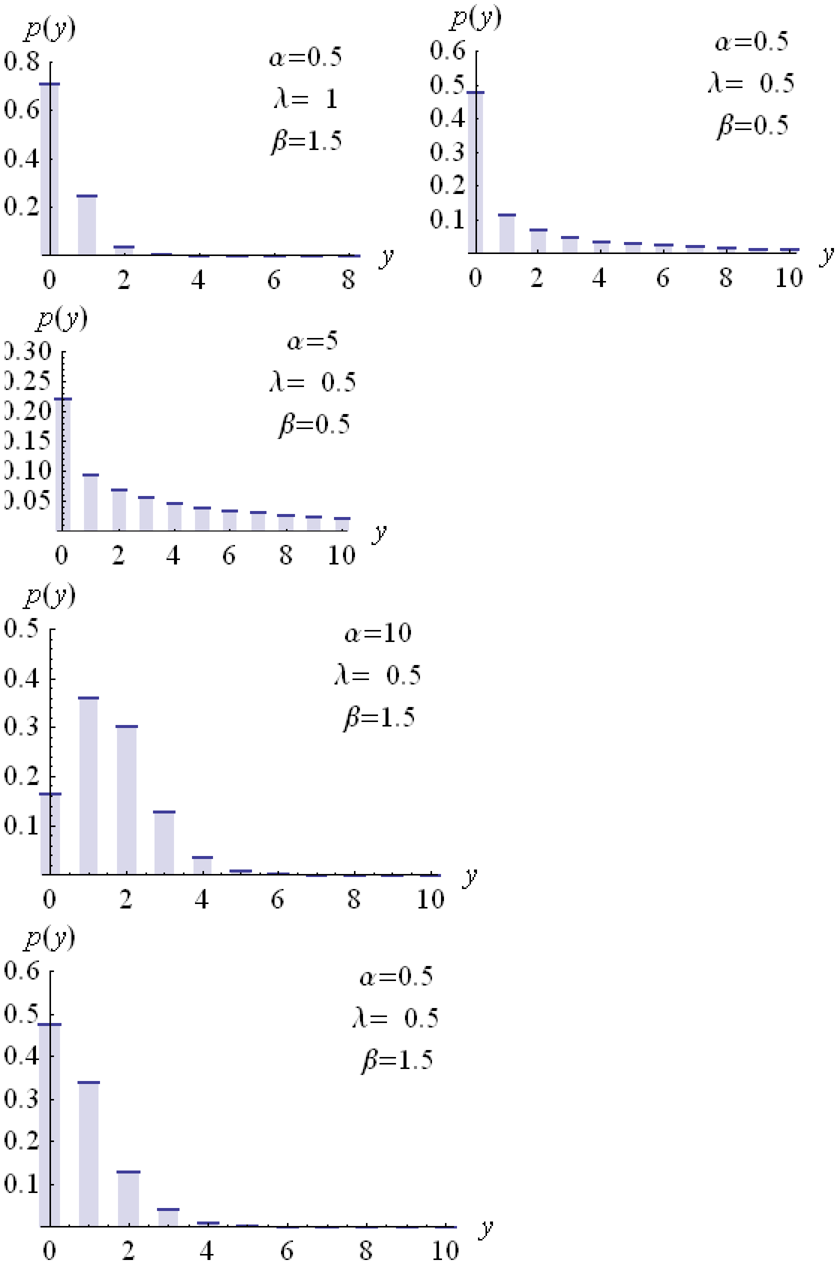
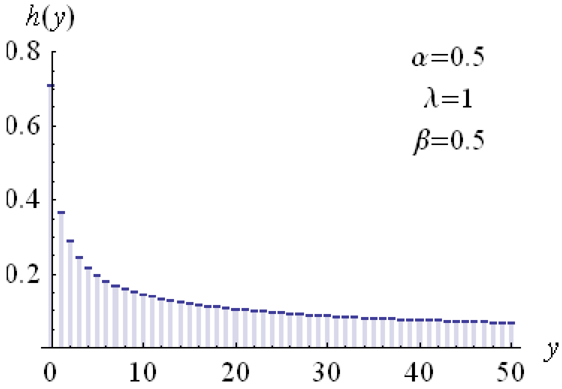
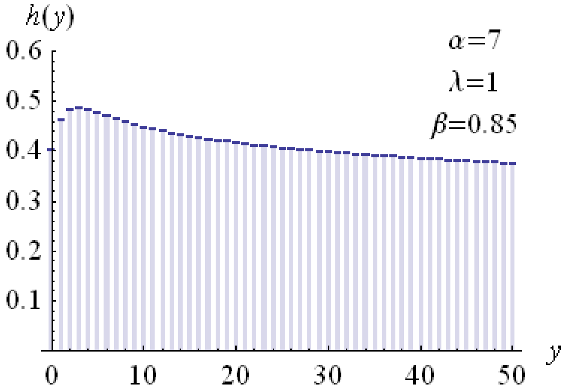


Figure 1. Plots of the probability mass function of DAPW for selected parameter values


h(y)
α=0.5
λ=1
β=0.5
0.8
0.6
0.4
0.2
0
10
20
30
40
50
y


h(y)
α=7
λ=1
β=0.85
0.6
0.5
0.4
0.3
0.2
0.1
0
10
20
30
40
50
y

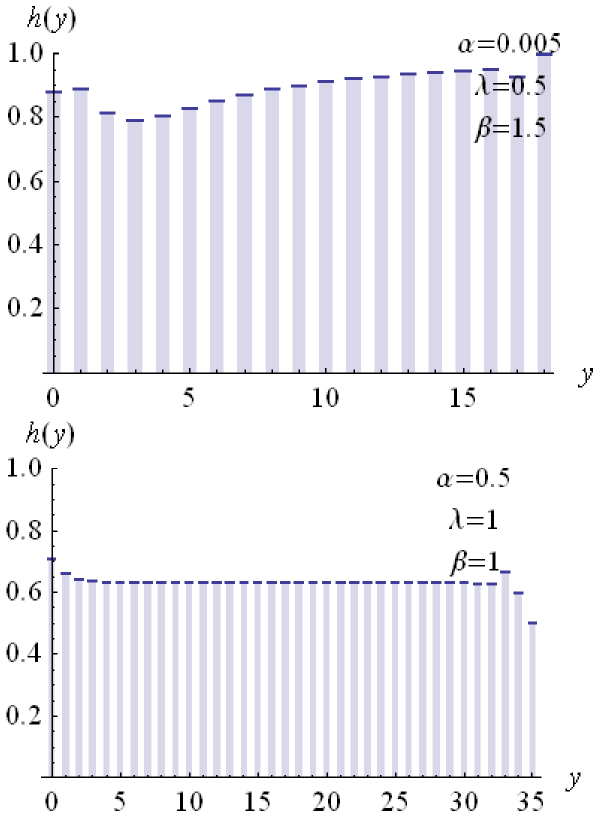


Figure 2. Plots of the hazard rate function of DAPW for selected parameter values

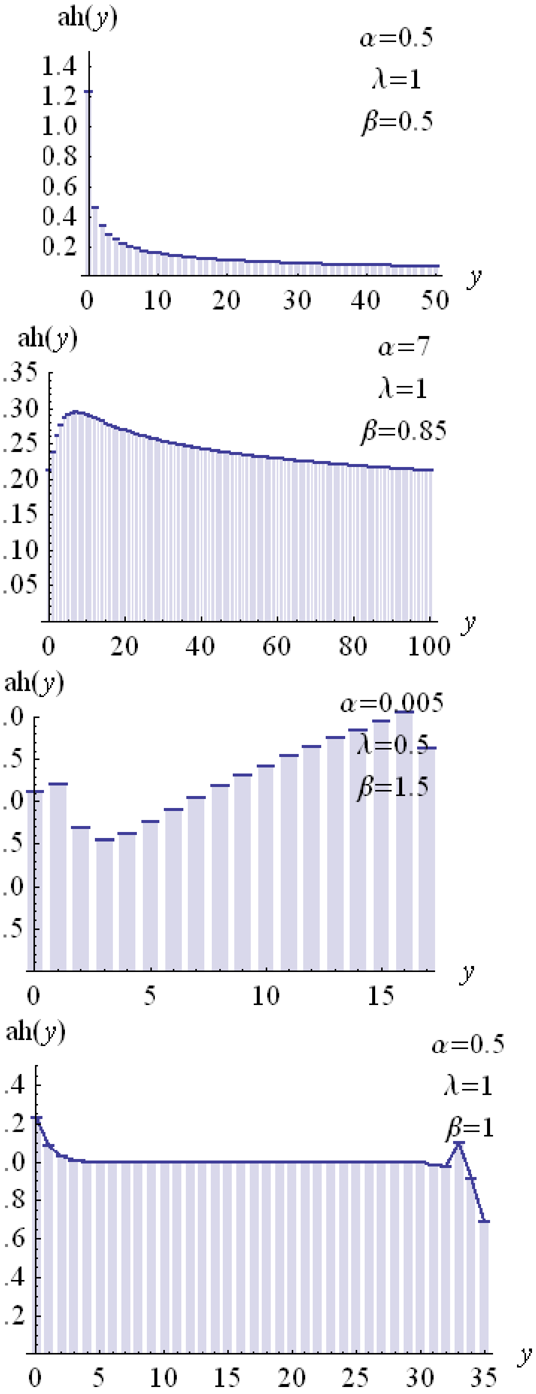


Figure 3. Plots of the alternative hazard rate function of DAPW for selected parameter values

4.4 Quantile function

The p –th quantile y_p of $F_{DAPW}(x)$, for $\alpha \neq 1$, can be obtained as

$$y_p = \left\{\frac{1}{log(p_2)} log\left[1 - \left(\frac{log[(\alpha-1)p+1]}{\log(\alpha)}\right)\right]\right\}^{1/\beta} - 1 \quad , \quad 0 < p < 1,$$

$$p_2 = e^{-\lambda}, 0 < p_2 < 1, \qquad (41)$$

Hence, the median can be obtained as follows:

$$y_{0.5} = \left\{\frac{1}{log(p_2)} log\left[1 - \left(\frac{log[(\alpha-1)0.5+1]}{\log(\alpha)}\right)\right]\right\}^{1/\beta} - 1 \quad , \qquad (42)$$

4.5 Mean, variance, skewness, and kurtosis

The mean (μ) of $\text{DAPW}(\alpha, p_2, \beta)$ distribution is given by

$$\mu = \sum_{y=0}^{\infty} y \left[(\alpha-1)^{-1}\left(\alpha^{1-p_2^{(y+1)^\beta}} - \alpha^{1-p_2^{y^\beta}}\right)\right], \qquad y = 0,1,2 \dots . \qquad (43)$$

and the variance is

$$\mu_2 = \sum_{x=0}^{\infty} X^2 \left[(\alpha-1)^{-1}\left(\alpha^{1-p_2^{(y+1)^\beta}} - \alpha^{1-p_2^{y^\beta}}\right)\right] - \left[\sum_{x=0}^{\infty} x \left[(\alpha-1)^{-1}\left(\alpha^{1-p_2^{(y+1)^\beta}} - \alpha^{1-p_2^{y^\beta}}\right)\right]\right]^2,$$

(44)

The skewness and kurtosis of the $\text{DAPW}(\alpha, p_2, \beta)$ distribution are given, respectively, by

$$\alpha_3 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \quad \text{and} \quad \alpha_4 = \frac{\mu_4}{\mu_2^2}, \text{ where } \mu_r = E(X-\mu)^r, \quad r = 1, 2, \dots, \quad (45)$$

The mean, median, variance, skewness and kurtosis of a $DAPW(\alpha, p_2, \beta)$ distribution for different values of α, p_2 and β are calculated numerically in Table 2 using (42) - (45). From Table 2, one can observe that depending on the values of the parameters, the mean of the distribution can be smaller or greater than the variance. Hence DAPW distribution models are appropriate for modeling both over and under dispersed data.

Table 2. The mean, median, variance, skewness and kurtosis of $\mathrm{DAPW}(\alpha, p_2 = e^{-\lambda}, \beta)$
for different values of the parameters

α, λ, β	Mean	Median	Variance	Skewness	Kurtosis
5 0.5 0.5 1 1.5	13.0282 2.3624 1.3702	5.0000 2.0000 1.0000	554.4120 5.3734 1.1600	5.1031 1.5483 0.7198	53.5341 6.6486 3.5277
5 1 0.5 1 1.5	2.9626 0.9571 0.6838	1.0000 1.0000 1.0000	34.1988 1.3291 0.4917	5.1760 1.6127 0.7588	54.6764 6.7605 3.2931
5 2 0.5 1 1.5	0.5306 0.2867 0.2517	0.0000 0.0000 0.0000	1.9521 0.3019 0.2026	5.6708 2.0609 1.3806	62.9851 7.9757 3.5556
10 0.5 0.5 1 1.5	15.6160 2.7187 1.5447	7.0000 2.0000 1.0000	661.1840 5.8082 1.1822	4.9609 1.4095 0.6180	45.8728 6.1198 3.4339
10 1 0.5 1 1.5	3.5877 1.1256 0.7911	1.0000 1.0000 0.0000	140.938 1.4588 0.5097	4.7400 1.4379 0.6069	46.5684 6.1003 3.1920
10 2 0.5 1 1.5	0.6569 0.3506 0.3064	0.0000 0.0000 0.0000	2.3782 0.3510 0.2306	5.1150 1.7644 1.0871	52.2084 6.608 2.8132

4.6 Mean residual life

The MRL of $\mathrm{DAPW}(\alpha, p_2, \beta)$

$$m(y_0) = \frac{\sum_{k=x_0+1}^{\infty}\left(1-\alpha^{-p_2^{k^\beta}}\right)}{\left(1-\alpha^{-p_2^{y_0^\beta}}\right)}, \qquad (46)$$

4.7 Order statistics

From (22), the cdf of the i^{th} order statistic for a random sample

$$F_{iDAPW}(y) = \sum_{r=i}^{n} \binom{n}{r} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left[\frac{\alpha^{1-p_2^{(y+1)^\beta}} - 1}{\alpha - 1}\right]^{r+j} , y = 0,1,2 \dots ; \quad \alpha \neq 1 .$$

(47)

From (27) and (28)

The pmf of the first order statistic

$$F_{DAPW}(Y_{1:n} = y) = \left[\left(\frac{\alpha}{\alpha-1}\right)\left(1 - \alpha^{-p_2^{y^\beta}}\right)\right]^n - \left[\left(\frac{\alpha}{\alpha-1}\right)\left(1 - \alpha^{-p_2^{(y+1)^\beta}}\right)\right]^n , y = 0,1,2 \dots ; \quad \alpha \neq 1. \quad (48)$$

The pmf of the largest order statistic

$$P(Y_{n:n} = y) = \left[\frac{\alpha^{1-p_2^{(y+1)^\beta}} - 1}{\alpha - 1}\right]^n - \left[\frac{\alpha^{1-p_2^{(y)^\beta}} - 1}{\alpha - 1}\right]^n , y = 0,1,2 \dots ; \quad \alpha \neq 1. \quad (49)$$

5. Maximum Likelihood Estimation

This section is devoted to estimate the vector of parameters, $\underline{\varphi} = (\alpha, p_2, \beta)$, sf, hrf and ahrf of the DAPW$(\underline{\varphi})$ distribution, also confidence intervals of the parameters α, p_2, β, sf, hrf and ahrf are derived.

Suppose that $Y_1, Y_2, \dots, Y_n$ is a sample of size n obtained from a life-test whose lifetimes have a DAPW $(\underline{\varphi})$ distribution. Then, the likelihood function is

$$L\left(\underline{\varphi}; \underline{y}\right) \propto \left\{\prod_{i=1}^{r} p\left(y_{(i)}\right)\right\}\left[S\left(y_{(r)}\right)\right]^{n-r},$$

(50)

Substituting (35) and (37) in (50). Hence

$$L\left(\underline{\varphi}; \underline{y}\right) \propto \left[\prod_{i=1}^{r} \left(\frac{\alpha}{\alpha-1}\right)\left(\alpha^{-p_2^{(y_i+1)^\beta}} - \alpha^{-p_2^{y_i^\beta}}\right)\right]\left[\left(\frac{\alpha}{\alpha-1}\right)\left(1 - \alpha^{-p_2^{y_{(r)}^\beta}}\right)\right]^{n-r} . \quad (51)$$

The ML estimator of $\underline{\varphi} = (\alpha, p_2, \beta)$ are obtained by maximizing the logarithm of the likelihood function, denoted by ℓ which can be written in the form:

$$\ell \equiv lnL\left(\underline{\varphi};\underline{y}\right)$$
$$\propto ln\left[\prod_{i=1}^{r}\left(\frac{\alpha}{\alpha-1}\right)\left(\alpha^{-p_2^{(y_i+1)^\beta}}-\alpha^{-p_2^{y_i^\beta}}\right)\right]+(n-r)\,ln\left(\frac{\alpha}{\alpha-1}\right)\left(1-\alpha^{-p_2^{y_{(r)}^\beta}}\right)$$
$$=\sum_{i=1}^{r} ln\left(\frac{\alpha}{\alpha-1}\right)\left(\alpha^{-p_2^{(y_i+1)^\beta}}-\alpha^{-p_2^{y_i^\beta}}\right)+(n-r)\,ln\left(\frac{\alpha}{\alpha-1}\right)\left(1-\alpha^{-p_2^{y_{(r)}^\beta}}\right)$$
$$=nln\left(\frac{\alpha}{\alpha-1}\right)+\sum_{i=1}^{r} ln\left(\alpha^{-p_2^{(y_i+1)^\beta}}-\alpha^{-p_2^{y_i^\beta}}\right)+(n-r)ln\left(1-\alpha^{-p_2^{y_{(r)}^\beta}}\right).$$

The ML estimators can be obtained setting the partial first derivatives of ℓ with respect to α, p_2 and β, respectively, to zeros. The system of non-linear equations can be solved numerically using the Newton-Raphson method, to obtain the ML estimators $\hat{\alpha}, \hat{p}_2$ and $\hat{\beta}$. The ML estimators $\hat{\alpha}, \hat{p}_2$ and $\hat{\beta}$ have an asymptotic variance-covariance matrix defined by inverting the information matrix.
Also, the ML estimators of the sf, hrf and ahrf can be derived using the invariance property of the ML estimators based on (37)-(39), respectively.
The asymptotic variance-covariance matrix of the estimators $\hat{\alpha}, \hat{p}_2$ and $\hat{\beta}$ are obtained depending on the inverse asymptotic Fisher information matrix $\tilde{I}$ using the second derivatives of the logarithm of the likelihood function.

The asymptotic Fisher information matrix can be written as follows:

$$\tilde{I} \approx -\left[\frac{\partial^2 \ell}{\partial \varphi_i \varphi_j}\right], \quad i,j = 1,2,3,$$
where $\varphi_1 = \alpha, \varphi_2 = p_2$ and $\varphi_3 = \beta$.

6. Numerical Illustration

This section aims to investigate the precision of the theoretical results based on simulated and real data through some measurements of accuracy; to study the precision and variation of the ML estimates.

6.1 Simulation

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates based on generated data from the DAPW distribution. The ML averages of the parameters, sf, hrf and ahrf based on complete sample and Type II censoring are computed. Moreover, credible intervals of the parameters, sf, hrf and ahrf are calculated. The simulation study is performed using Mathematica 9.

Table 3 shows the averages, *relative absolute biases* (RABs), *Relative errors* (REs), variances, sf, hrf and ahrf estimates, also 95% confidence intervals where the initial values for the parameters are $\alpha = 5$, $\lambda = 1.3$, $\beta = 0.5$ under two levels of $\frac{r}{n} \times 100$ percentage of uncensored observations Type II censoring 80% and 100%. Table 4 and 5 displays the same computational results under complete sample, but for different true parameter values from the DAPW distribution for different samples of size where (*n*=50 and 100) and number *of replications*, *NR* = 1000.

The RABs, REs, ERs and variances of the ML estimates of the parameters, sf, hrf and ahrf are computed as follows:

1) Averages = $\frac{\sum_{i=1}^{NR} \text{estimates}}{NR}$

2) ABs (estimate) = $\frac{|\text{bias (estimate)}|}{\text{true value}}$,

3) Es =$\frac{\text{ER(estimate)}}{\text{true value}}$,

4) Variances (estimate) = $\text{ER(estimate)} - \text{bias}^2\ \text{(estimate)},$

Table 3. ML averages, relative absolute biases, relative errors, variances of ML estimates, 95%confidence intervals of the parameters from DAPW distribution for different sample sizes *n*, censoring size *r*, $y_0 = 1$ and the number of replications *NR*= 1000 ($\alpha = 5$, $\lambda = -log p_2 = 1.3$, $\beta = 0.5$)

n	*r*	Parameters	Averages	RABs	REs	Variances	UL	LL	Length
100	100 80	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	5.1587 1.0710 0.6462 0.6789 0.3749 0.4997	0.0318 0.2860 0.2924 0.2713 0.2035 0.2146	0.0438 0.4544 0.4502 0.3880 0.4272 0.4155	0.0225 0.2806 0.0293 0.0219 0.0313 0.0512	5.4526 2.1091 0.9817 0.9692 0.7215 0.9433	4.8649 0.0328 0.3107 0.3886 0.0284 0.0560	0.5877 2.0763 0.6710 0.5806 0.6931 0.8873
	100 100	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	5.0734 1.2933 0.5596 0.6087 0.4355 0.5773	0.0146 0.1378 0.1192 0.1399 0.0748 0.0926	0.0255 0.2028 0.1780 0.1871 0.1723 0.1763	0.0109 0.0498 0.0043 0.0044 0.0054 0.0092	5.2781 1.7308 0.6891 0.7388 0.5776 0.76377	4.8688 0.8557 0.4300 0.4786 0.2932 0.3907	0.4092 0.8751 0.2590 0.2601 0.2844 0.3730
200	200 160	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	5.1192 1.1281 0.6343 0.6622 0.3945 0.5285	0.0238 0.2479 0.2687 0.2401 0.1618 0.1691	0.0361 0.4164 0.4128 0.3578 0.3852 0.3759	0.0185 0.2519 0.0245 0.0200 0.0270 0.0456	5.3859 2.1119 0.9415 0.9398 0.7170 0.9472	4.8526 0.1443 0.3272 0.3846 0.0719 0.1098	0.5332 1.9675 0.6143 0.5552 0.6450 0.8373
	200 200	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	5.0737 1.2817 0.5618 0.6118 0.4311 0.5716	0.0138 0.1363 0.1159 0.1379 0.0747 0.0923	0.0220 0.2017 0.1723 0.1852 0.1713 0.1759	0.0072 0.0497 0.0040 0.0043 0.0053 0.0091	5.2367 1.7326 0.6829 0.7371 0.5787 0.7647	4.9022 0.8582 0.4329 0.4781 0.2922 0.3900	0.3345 0.8744 0.2499 0.2589 0.2865 0.3747

Table 4. ML averages, relative absolute biases, relative errors, variances of ML estimates, 95%confidence intervals of the parameters from DAPW distribution for sample sizes *n=50*, for different values of the parameters, $y_0 = 1$ and the number of replications *NR*= 1000

n	*r*	Parameters	Averages	RABs	REs	Variances	UL	LL	Length
50	$\alpha = 0.5$ $\lambda = 1$ $\beta = 0.5$	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	0.4943 0.9711 0.5090 0.4189 0.4424 0.5842	0.0115 0.0289 0.0181 0.0282 0.0071 0.0095	0.0346 0.0803 0.0491 0.0770 0.0199 0.0266	0.0003 0.0056 0.0005 0.0009 0.0001 0.0002	0.5262 1.1181 0.5537 0.4762 0.4587 0.6130	0.4623 0.8242 0.4643 0.3617 0.4261 0.5555	0.0638 0.2939 0.0894 0.1144 0.0326 0.0575
50	$\alpha = 0.5$ $\lambda = 1$ $\beta = 1.9$	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	0.7022 1.2908 1.2806 0.5353 0.7760 1.6588	0.4045 0.2908 0.3260 0.2344 0.1192 0.2209	0.5207 0.3536 0.4170 0.3051 0.2044 0.3322	0.0269 0.0404 0.2442 0.0186 0.0214 0.2791	1.0236 1.6848 2.2491 0.8030 1.0628 2.6942	0.3809 0.8970 0.3122 0.2677 0.4893 0.6235	0.6427 0.7878 1.9369 0.5353 0.5735 2.0707
50	$\alpha = 1.5$ $\lambda = 1$ $\beta = 0.5$	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	1.4877 0.8946 0.5425 0.4581 0.3155 0.3791	0.0340 0.1054 0.0851 0.0938 0.0186 0.0225	0.0524 0.1308 0.1235 0.1268 0.0287 0.0346	0.0038 0.0060 0.0020 0.0013 0.00005 0.0001	1.6081 1.0465 0.6303 0.5282 0.3293 0.3992	1.3673 0.7427 0.4547 0.3880 0.3018 0.3591	0.2409 0.3038 0.1756 0.1402 0.0275 0.0400
50	$\alpha = 3$ $\lambda = 1$ $\beta = 0.5$	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	3.1327 0.8415 0.5788 0.5715 0.2803 0.3291	0.0442 0.1585 0.1577 0.1460 0.0498 0.0583	0.1239 0.2003 0.2395 0.1970 0.0746 0.0859	0.1204 0.0150 0.0081 0.0044 0.0003 0.0005	3.8130 1.0818 0.7556 0.7009 0.3124 0.3724	2.4525 0.6013 0.4021 0.4421 0.2482 0.2859	1.3605 0.4806 0.3535 0.2587 0.0642 0.0865

Table 5.

ML averages, relative absolute biases, relative errors, variances of ML estimates, 95%confidence intervals of the parameters from DAPW distribution for sample sizes *n* =100, for different values of the parameters, $y_0 = 1$ and the number of replications *NR*= 1000

n	*r*	Parameters	Averages	RABs	REs	UL	LL	Length
100	$\alpha = 0.5$ $\lambda = 1$ $\beta = 0.5$	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	0.4977 0.9891 0.5032 0.4117 0.4443 0.5876	0.0047 0.0109 0.0065 0.0105 0.0029 0.0038	0.0210 0.0490 0.0290 0.0471 0.0130 0.0172	0.5178 1.0828 0.5310 0.4483 0.4553 0.6070	0.4775 0.8954 0.4754 0.3751 0.4333 0.5681	0.0402 0.1874 0.0555 0.0733 0.0221 0.0389
100	$\alpha = 0.5$ $\lambda = 1$ $\beta = 1.9$	α λ β $R(y_0)$ $h(y_0)$ $ah\ (y_0)$	0.6829 1.2730 1.3409 0.5512 0.7959 1.7262	0.3658 0.2730 0.2943 0.2118 0.0967 0.1893	0.4742 0.3368 0.3772 0.2794 0.1769 0.2908	0.9786 1.6595 2.21985 0.8009 1.0519 2.6473	0.3871 0.8865 0.4618 0.3014 0.5399 0.8049	0.5914 0.7729 1.7579 0.4995 0.5119 1.8424

100	$\alpha = 1.5$ $\lambda = 1$ $\beta = 0.5$	α	1.4818	0.0121	0.0419	1.5997	1.3640	0.2357
		λ	0.8936	0.1064	0.1271	1.0299	0.7572	0.2727
		β	0.5408	0.0816	0.1171	0.6230	0.4586	0.1645
		$R(y_0)$	0.4579	0.1014	0.1282	0.5217	0.3940	0.1277
		$h(y_0)$	0.3147	0.0244	0.0311	0.3269	0.3025	0.0244
		$ah\ (y_0)$	0.3779	0.0296	0.0376	0.3957	0.3602	0.0355
100	$\alpha = 3$ $\lambda = 1$ $\beta = 0.5$	α	3.0415	0.0138	0.0562	3.3623	2.7207	0.6415
		λ	0.8756	0.1244	0.1516	1.0453	0.7057	0.3396
		β	0.5529	0.1057	0.1483	0.6547	0.4510	0.2037
		$R(y_0)$	0.5530	0.1089	0.1412	0.6409	0.4651	0.1758
		$h(y_0)$	0.2828	0.0410	0.0622	0.3099	0.2557	0.0541
		$ah\ (y_0)$	0.3326	0.0482	0.0715	0.3688	0.2964	0.0723

6.2 Application

In this section, the flexibility of the DAPW distribution is illustrated through using three real data sets.

Application 1:

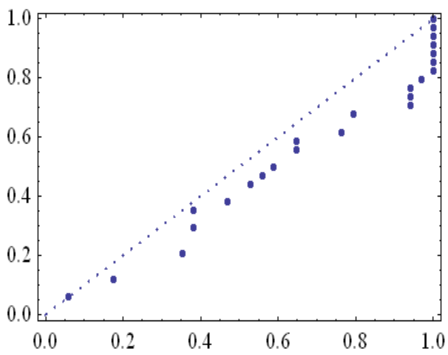
The first application is the vinyl chloride data obtained from clean upgrading, monitoring wells in mg/L; this data set was used by Bhaumik *et al.* (2009). The data is:
5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.10, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

Application 2:

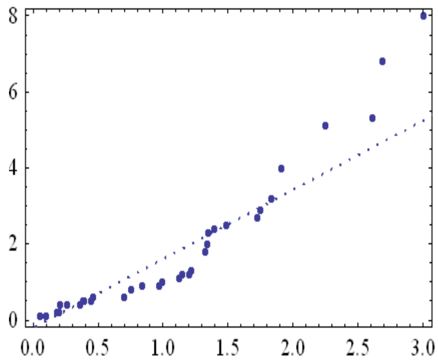
The second data set contains fifty observations of lifetime presented by Aarset (1987).
The data set is 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 1, 1, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Application 3:

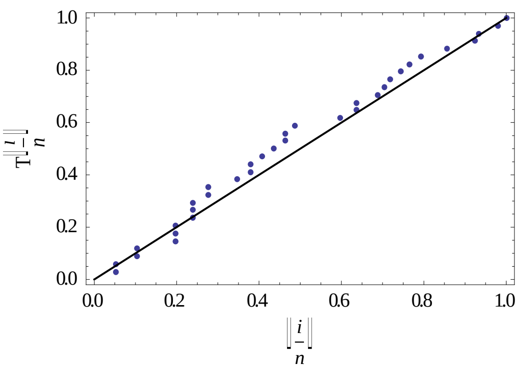
The third data set is given by Murthy *et al.* (2004). It refers to the time between failures for 30 repairable objects. The data is 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 1.97, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 1.86, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, and 1.17.



P-plot for the first data set

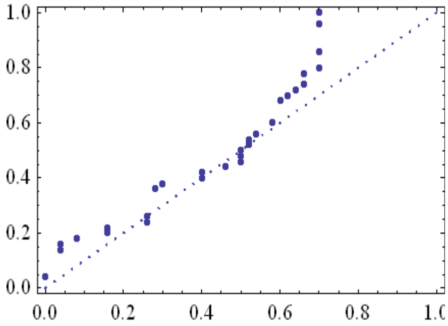


QQ-plot for the first data set

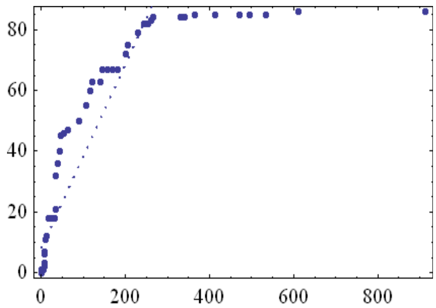


TTT-plot for the first data set

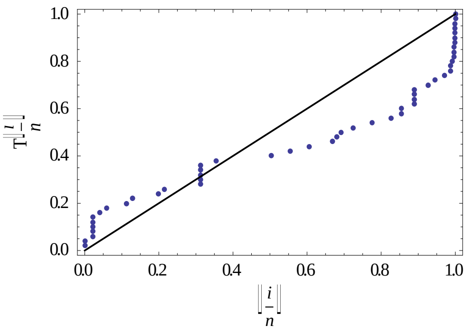
Figure 4: PP-plot, QQ-plot and TTT-plot of the DAPW distribution for the first data set



PP-plot for the second data set

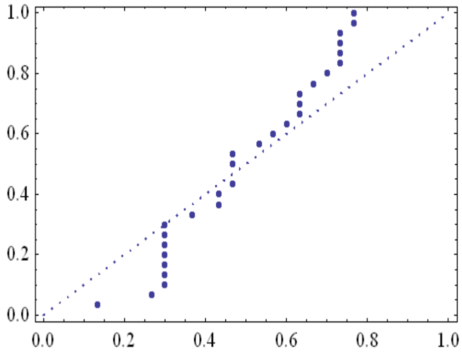


QQ-plot for the second data set

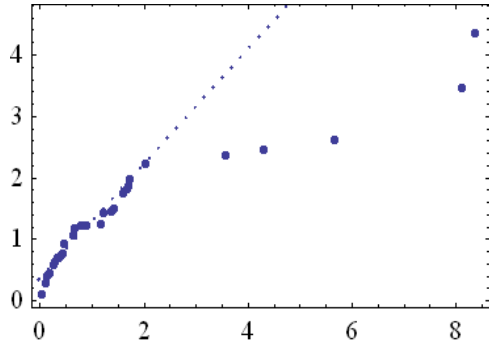


TTT-plot for the second data set

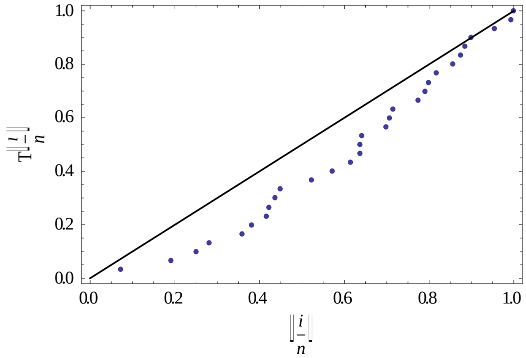
Figure 5: PP-plot, QQ-plot and TTT-plot of the DAPW distribution for the second data set



PP-plot for the third data set



QQ-plot for the third data set



TTT-plot for the third data set

Figure 6: PP-plot, QQ-plot and TTT-plot of the DAPW distribution for the third data set

Table 6. Parameter estimates and goodness of fit for various models fitted for the second data

Models	Estimates	SEs	p-value	AIC	BIC	AICC	HQIC
DAPW	$\alpha = 5$ $\lambda = 0.9$ $\beta = 1.2$	0.31 0.26 0.25	0.65	171.95	177.95	182.53	178.75
DAPR	$\alpha = 3.25$ $\lambda = 0.42$	0.24 0.28	0.10	391.08	395.08	398.13	395.47
DAPE	$\alpha = 2.97$ $\lambda = 0.91$	0.24 0.26	0.29	239.63	242.63	246.68	244.01
DAPWO	$\alpha = 1.4$ $\beta = 0.2$	0.24 0.26	0.05	176.34	180.34	183.39	180.73

Table 7. Parameter estimates and goodness of fit for various models fitted for the third data

Models	Estimates	SEs	p-value	AIC	BIC	AICC	HQIC
DAPW	$\alpha = 2.40$ $\lambda = 1.15$ $\beta = 0.82$	0.19 0.21 0.23	0.59	182.97	188.97	193.17	189.89
DAPR	$\alpha = 5.92$ $\lambda = 0.39$	0.37 0.25	0.59	278.21	288.21	285.02	282.66
DAPE	$\alpha = 8.60$ $\lambda = 1.00$	0.47 0.22	0.07	242.63	246.63	249.43	247.08
DAPWO	$\alpha = 1.08$ $\beta = 1.02$	0.22 0.22	0.06	190.23	194.23	197.28	194.62

Kolmogorov-Smirnov (K-S) goodness of fit test is applied to check the validity of the fitted model. The p-values are 0.1069, 0.65 and 0.59, respectively. It shows that DAPW fits the data very well.

Figures 4-6 present the PP and QQ plots and TTT plot for the three real data sets, which indicates that the DAPW distribution provides better fit to the data sets. The TTT plot for the first and second real data sets which are displayed in Figures 4 and 5 provide evidence that the first and second data sets possesses bathtub hrf, but the TTT plot of the third real data set in Figure 6 indicates that the hrf is decreasing function.

The proposed distribution: DAPW distribution is compared to other distributions which are considered sub-models DAPW distribution such as *discrete alpha power one parameter Weibull* (DAPWO) distribution, *discrete alpha power Rayleigh* (DAPR) distribution and *discrete alpha power exponential distribution* (DAPE).

To verify which distribution fits better to the real data sets, the values of the *Akaike Information Criterion* (AIC), *Akaike Information Criterion with correction* (AICC), *Bayesian Information Criterion* (BIC) and *Hannon-Quinn Information Criterion* (HQIC) are calculated for second and third real data sets. The best distribution corresponds to the lowest values of AIC, AICC, BIC and HQIC, also the highest p-value,

where $AIC = -2\log L + 2k$, $AICC = AIC + \frac{2k(k+1)}{n-k-1}$, $BIC = -2\log L + k\log n$ and

$HQIC = -2\log L + 2k\log(\log(n))$, where k is the number of the parameters and n is the sample size and L is the maximized value of the likelihood function for the estimated model. Tables 6 and 7 display the values of p-value, AIC, AICC, BIC and HQIC for the first and third data sets.

7. Conclusion

In this paper, a family of discrete distributions is proposed. Generalizations of discrete uniform, discrete exponential, discrete Rayleigh and discrete Weibull are obtained using this family. Also, many other discrete distributions can be obtained as sub models. As a particular case, discrete alpha power Weibull distribution is introduced. Some of its properties are studied. The ML estimators for the model parameters are derived. The discrete alpha power Weibull distribution appears to be more suitable for modeling real data sets and is a better alternative to some distributions.

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