Exponentiated Inverse Nadarajah-Haghighi Distribution: Properties and Applications

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Abstract

In this paper, we introduced a new three-parameter distribution named exponentiated inverted Nadarajah-Haghighi (EINH) distribution. This distribution is generalized of inverse Nadarajah-Haghighi distribution. It can be also considered as an extension of the inverse exponential distribution. Some of the mathematical properties of the new distribution are studied. The quantile function, density and distribution functions, moments, generating function and order statistics are obtained. The model parameters are estimated using the maximum likelihood method. Two real applications are used to show the flexibility of the EINH distribution.

Keywords: Nadarajah-Haghighi distribution; maximum likelihood estimation; moment generating function; hazard rate function; order statistics

1. Introduction

The one-parameter exponential distribution is distinguished from other distributions by its simple form. It also is one of the most important distributions used in the modeling of lifetimes data. A new extension of the exponential distribution called the Nadarajah-Haghighi distribution has been proposed by Nadarajah and Haghighi (2011). In previous years, many authors proposed new extensions for Nadarajah-Haghighi distribution such as Lemonte (2013) proposed exponentiated Nadarajah-Haghighi distribution, Bourguignon et al. (2015) proposed gamma Nadarajah-Haghighi distribution, VedoVatto et al. (2016) proposed exponentiated generalized Nadarajah-Haghighi, (Dias et al. 2016) proposed beta Nadarajah-Haghighi, Lima (2015) proposed kumaraswamy Nadarajah-Haghighi , (Yousof and Korkmaz 2017) proposed Topp-Leone Nadarajah-Haghighi distribution and Yousof et al. (2017) proposed the odd Lindley Nadarajah-Haghighi distribution.

Recently, Tahir et al. (2018) introduced the inverse Nadarajah-Haghighi distribution (INH). Which has the cumulative density function (cdf), probability density function (pdf) and reliability function as follow

$$F(x) = \exp\left\{1 - \left(1 + \frac{\lambda}{x}\right)^{\alpha}\right\}, \qquad \lambda, \alpha > 0, \ x > 0, \tag{1}$$

$$f(x) = \lambda \alpha x^{-2} \left(1 + \frac{\lambda}{x}\right)^{\alpha - 1} \exp\left\{1 - \left(1 + \frac{\lambda}{x}\right)^{\alpha}\right\} \tag{2}$$

and

$$R(x) = 1 - \exp\left\{1 - \left(1 + \frac{\lambda}{x}\right)^{\alpha}\right\}, \tag{3}$$

respectively, where α and λ are shape and scale parameters, respectively. The exponential distribution is a special case of (1) when $\alpha = 1$. This paper aims to introduce a generalization of the inverse Nadarajah-Haghighi distribution called exponentiated inverse Nadarajah-

Haghighi (EINH) distribution and studies its mathematical properties. The rest of this paper is ordered as follows. In section 2, we introduce the exponentiated inverse Nadarajah-Haghighi (EINH) distribution. In section 3, we derive some of the mathematical properties of EINH distribution. The maximum likelihood method is used to estimate the model parameters in Section 4. A real data set is used to illustrate the usefulness of the EINH distribution in section 5. The concluding comments are given at the end.

2. Exponentiated Inverse Nadarajah –Haghighi Distribution

Nadarajah and Kotz (2006) suggested the exponentiated family of distributions by using reliability function as follows

$$F(x) = 1 - \left(R(x)\right)^{\alpha} \tag{4}$$

Now we can apply eq (3) in eq (4) to get the cdf of EINH as follows:

$$F(x) = 1 - \left(1 - \exp\left\{1 - \left(1 + \frac{\lambda}{x}\right)^{\alpha}\right\}\right)^{\beta}, \qquad \lambda, \alpha, \beta > 0, \ \ x > 0 \tag{5}$$

and by differentiation (5), we get the pdf as follows

$$f(x) = \lambda\alpha\beta x^{-2}\left(1 + \frac{\lambda}{x}\right)^{\alpha-1} \exp\left\{1 - \left(1 + \frac{\lambda}{x}\right)^{\alpha}\right\}\left(1 - \exp\left\{1 - \left(1 + \frac{\lambda}{x}\right)^{\alpha}\right\}\right)^{\beta-1} \tag{6}$$

where λ is scale parameter and α, β are shape parameters. This model has as special cases, the inverse exponential (IE) distribution by Keller et al. (1982) when $\alpha = \beta = 1$ and the inverse Nadarajah –Haghighi distribution by Tahir et al. (2018) when $\beta = 1$. Plots of pdf and cdf of the

EINH distribution for some values of parameters are shown in Figures1 and 2, respectively.

2.1. Survival and Hazard Rate Function

The survival $s(t)$ and the hazard rate $h(t)$ functions of the EINH distribution are given, respectively by

$$s(t) = 1 - F(x) = \left(1 - \exp\left\{1 - \left(1 + \frac{\lambda}{t}\right)^{\alpha}\right\}\right)^{\beta} \quad , \quad t > 0 \tag{7}$$

and

$$h(t) = \frac{f(t)}{S(t)} = \lambda\alpha\beta x^{-2}\left(1 + \frac{\lambda}{t}\right)^{\alpha-1} \exp\left\{1 - \left(1 + \frac{\lambda}{t}\right)^{\alpha}\right\}\left(1 - \exp\left\{1 - \left(1 + \frac{\lambda}{t}\right)^{\alpha}\right\}\right)^{-1} \quad , \quad t > 0 \tag{8}$$

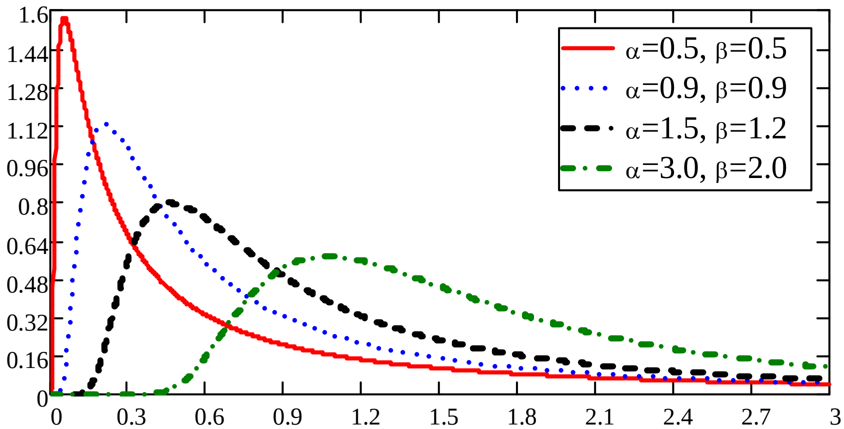


Fig. 1: Some possible shapes of the EINH density function with $\lambda = 0.5$

3. The Statistical Properties

In this section, some of the statistical properties of EINH distribution including the quantile function, random variables generation function, moments, moment generating function, skewness, kurtosis and order statistics are derived.

3.1 Expansions for the cumulative and density functions

The expansion for the cumulative distribution function of EINH distribution can be derived by using the generalized binomial theorem. For any real number $r > 0$ and $|z| < 1$ the binomial expansion is

$$(1-z)^r = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} z^i \tag{9}$$

where $\binom{r}{i} = \frac{r(r-1)\dots(r-i+1)}{i!}$.

Using the binomial expansion (9) in equation (5), we get the cdf as a power series expansion as follows

$$F(x) = 1 - \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} \left(\exp\left\{ 1 - \left(1 + \frac{\lambda}{x}\right)^{\alpha} \right\} \right)^i , \tag{10}$$

using the binomial expansion (9), again in the last term of (10), we get the expansion of cdf as follow

$$F(x) = 1 - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{b}{i} \binom{ai}{j} e^{j\left(1-\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta}\right)} . \tag{11}$$

Differentiating (11) with respect to x gives the expansion of pdf as follow

$$f_{kgpw}(x) = \alpha\lambda\beta x^{-2} \left(1 + \frac{\lambda}{x}\right)^{\alpha-1} \sum_{i=0}^{\beta-1} (-1)^i \binom{\beta-1}{i} \exp\left\{ (i+1)\left(1 - \left(1 + \frac{\lambda}{x}\right)^{\alpha}\right) \right\} . \tag{12}$$

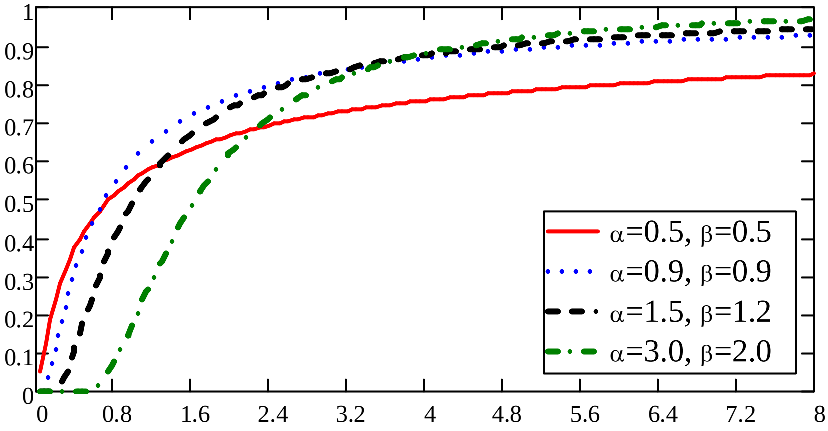


Fig. 2: Some possible shapes of the EINH cumulative density function with $\lambda = 0.5$.

3.2 Quantile function and simulation

The quantile function of EINH distribution can be obtained by inverting (5) as follow

$$Q(q) = \lambda\left[\left(1 - \ln\left\{1 - (1-q)^{1/\beta}\right\}\right)^{1/\alpha} - 1\right]^{-1}. \quad (13)$$

As a result, the median of $X \sim EINH$ distribution, is

$$Q(0.5) = \lambda\left[\left(1 - \ln\left\{1 - (0.5)^{1/\beta}\right\}\right)^{1/\alpha} - 1\right]^{-1} \quad (14)$$

Figure 3, shows the median of EINH distribution as a function in the parameter β.

The quantile function of EINH distribution can be also used to generate random variables of EINH distribution as following

$$X = \lambda\left[\left(1 - \ln\left\{1 - (1-u)^{1/\beta}\right\}\right)^{1/\alpha} - 1\right]^{-1} \quad (15)$$

where $X \sim EINH(\lambda, \alpha)$ and $U \sim (0,1)$.

3.3 Skewness and kurtosis

The Bowley's skewness measure based on quartiles ((Kenney and Keeping 1962)) is given by

$$Sk = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \qquad (16)$$

and the Moors' kurtosis measure based on octiles (Moors (1988)) is given by

$$Ku = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \qquad (17)$$

where Q(.) is given by (13). Figure 4, shows the behaviors of skewness and kurtosis of the EINH distribution as a function of the parameter β. The skewness and kurtosis are decreasing with β.

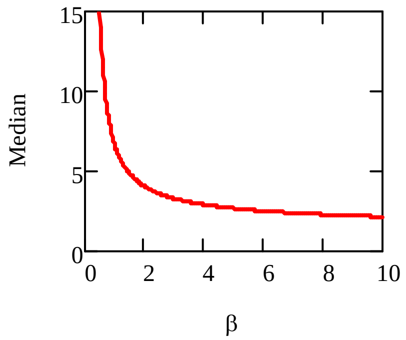


Figure 3: the graphical representation of the median as a function of the parameter β for $\alpha = \lambda = 2$.

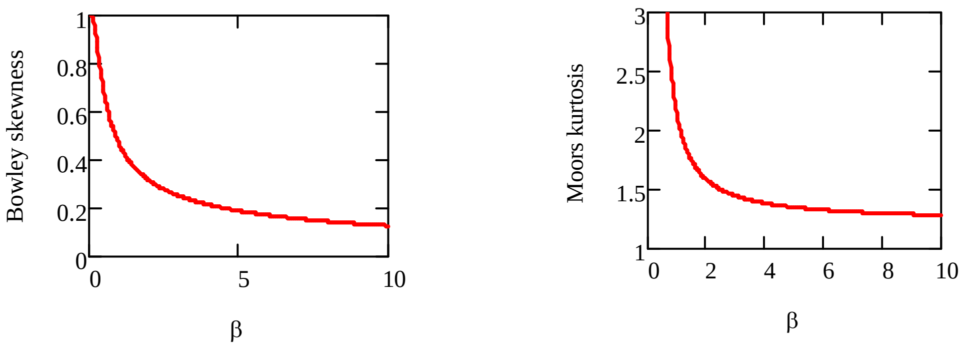


Figure 4: the graphical representation of the Bowley's skewness and Moors' kurtosis as a function of the parameter β for $\alpha = \lambda = 2$.

3.4 Moments and moment generating function

If x has the EINH distribution, the moments and moment generating function are given by the following theorem

Theorem 1. Let X have a EINH distribution, Then the rth ordinary moments of x for integer value of r/λ is

$$\mu_r' = \beta\lambda^r \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j+i-r}\binom{\beta-1}{i}\binom{-r}{j}\frac{e^{i+1}}{(i+1)^{j+\alpha/\alpha}}\Gamma\left[\frac{j+\alpha}{\alpha}, i+1\right], \quad r<1 \qquad (18)$$

the equation (18) only exists when r <1. This means that the first moment and other higher-order moments do not exist.

$$\alpha\lambda\beta x^{-2}\left(1+\frac{\lambda}{x}\right)^{\alpha-1}\sum_{i=0}^{\beta-1}(-1)^i\binom{\beta-1}{i}\exp\left\{(i+1)\left(1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right)\right\}$$

Proof. The r th moment of X is defined as follows

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f(x)dx \qquad (19)$$

Substituting (6) into (19) yields

$$\mu_r' = \alpha\lambda\beta \int_0^{\infty} x^{r-2}\left(1+\frac{\lambda}{x}\right)^{\alpha-1} \sum_{i=0}^{\beta-1}(-1)^i \binom{\beta-1}{i} \exp\left\{(i+1)\left(1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right)\right\} \qquad (20)$$

Let $v=\left(1+\frac{\lambda}{x}\right)^{\alpha}$, the above expression reduce to

$$\mu_r' = e^{(i+1)}\lambda^r \int_1^{\infty} \left(v^{\frac{1}{\alpha}}-1\right)^{-r} e^{-(i+1)v} du \qquad (21)$$

Then, by applying the binomial expansion of $\left(v^{\frac{1}{\alpha}}-1\right)^{-r} = \sum_{i=0}^{\infty}(-1)^{i-r}\binom{-r}{i} v^{\frac{i}{\alpha}}$ we get

$$\mu_r' = e^{(i+1)}\lambda^r \sum_{i=0}^{-r}(-1)^{i-r}\binom{-r}{i} \int_1^{\infty} v^{\frac{i}{\alpha}} e^{-(i+1)v} dv \qquad (22)$$

By integrating the incomplete gamma function we get the r *th* moment of x as follows

$$\mu_r' = \beta\lambda^r \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j+i-r}\binom{\beta-1}{i}\binom{-r}{j}\frac{e^{i+1}}{(i+1)^{j+\alpha/\alpha}}\Gamma\left[\frac{j+\alpha}{\alpha}, i+1\right], \qquad r<1$$

If $\alpha=1=\beta$, we get the moments of inverse exponential distribution as follows

$$\mu_r' = e\lambda^r \sum_{i=0}^{-r}(-1)^{i-r}\binom{-r}{i}\Gamma[i+1,1]$$

Theorem 2. If $X\sim EINH$ distribution, then for any integer value of ${}^{r}/_{\lambda}$ the moment generating function (mgf) is

$$M_x(t) = \beta\lambda^r \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j+i-r}\binom{\beta-1}{i}\binom{-r}{j}\frac{(t)^r}{r!}\frac{e^{i+1}}{(i+1)^{j+\alpha/\alpha}}\Gamma\left[\frac{j+\alpha}{\alpha}, i+1\right] \qquad (23)$$

Proof. The moment generating function is defined as follows

$$M_x(t) = \int_0^\infty e^{tx} f(x)dx$$

Using exponential function formula $e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$, we get

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \qquad (24)$$

Inserting (18) in equation (24) yields the mgf of EINH distribution as in (19).

3.5 Order statistics

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ are the order statistics of a random sample follows a continuous distribution with cdf $F(x)$ and pdf $f(x)$. Then the pdf of $X_{(k:n)}$ is

$$f_{k:n}(x) = \frac{n!}{(k-1)!\,(n-k)!} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}, \quad k = 1,2,\dots,n \qquad (25)$$

Therefore, the pdf of the k-th order statistics of the EINH distribution is

$$f_{k:n}(x) = \frac{n!}{(k-1)!\,(n-k)!} \lambda\alpha\beta x^{-2} \left(1+\frac{\lambda}{x}\right)^{\alpha-1} \exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\} \left\{1 - \left(1-\exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\}\right)^{\beta}\right\}^{k-1} \left(1-\exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\}\right)^{\beta(n-k)+\beta-1} \qquad (26)$$

if $k = 1$, the pdf of order statistics is

$$f_{1:n}(x) = n\lambda\alpha\beta x^{-2} \left(1+\frac{\lambda}{x}\right)^{\alpha-1} \exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\} \left(1-\exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\}\right)^{\beta n-1} \qquad (27)$$

and if $k = n$, the pdf of order statistics is

$$f_{n:n}(x) = n\lambda\alpha\beta x^{-2}\left(1+\frac{\lambda}{x}\right)^{\alpha-1}\exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\}\left(1-\exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\}\right)^{\beta-1}\left\{1-\left(1-\exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\}\right)^{\beta}\right\}^{n-1} \quad (28)$$

4. Maximum Likelihood Estimation

This section is dedicated to investigating the maximum likelihood estimation (???) and the approximate confidence intervals for the unknown parameters of EINH distribution. Let $x_1, x_1, \ldots, x_n$ is a random sample of size n from the EINH distribution. Then the likelihood function (LF) is

$$\mathcal{L} = (\alpha\lambda\beta)^n \prod_{i=1}^{n} x_i^{-2}\left(1+\frac{\lambda}{x_i}\right)^{\alpha-1}\exp\left\{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}\right\}\left(1-\exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\}\right)^{\beta-1}$$

and the log-likelihood function ($ln\mathcal{L}$) is

$$ln\mathcal{L} = nln(\alpha\lambda\beta) - 2\sum_{i=1}^{n} ln\, x_i + (\alpha-1)\sum_{i=1}^{n} ln\left(1+\frac{\lambda}{x_i}\right) + n - \sum_{i=1}^{n}\left(1+\frac{\lambda}{x_i}\right)^{\alpha} + (\beta-1)\sum_{i=1}^{n} ln\left(1-\exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}\right\}\right) \quad (29)$$

Then, the maximum likelihood estimators (MLE) of β is

$$\hat{\beta} = -\frac{n}{\sum_{i=1}^{n} ln\left[1-e^{1-\left(1+\frac{\hat{\lambda}}{x_i}\right)^{\hat{\alpha}}}\right]} \quad (30)$$

where $\hat{\alpha}$ and $\hat{\lambda}$ are the MLEs of the parameters α and λ, which can be obtained as a solution of the following non-linear equations

$$\frac{\partial ln\mathcal{L}}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}ln\left(1+\frac{\lambda}{x_i}\right)-\sum_{i=1}^{n}ln\left(1+\frac{\lambda}{x_i}\right)\left(1+\frac{\lambda}{x_i}\right)^{\alpha}-\left[\frac{n}{\sum_{i=1}^{n}ln\left[1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right]}+1\right]\sum_{i=1}^{n}\frac{e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\ln\left(1+\frac{\lambda}{x_i}\right)\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}{1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}}=0. \quad (31)$$

$$\frac{\partial ln\mathcal{L}}{\partial\lambda}=\frac{n}{\lambda}+(\alpha-1)\sum_{i=1}^{n}\frac{1}{\left(1+\frac{\lambda}{x_i}\right)x_i}-\alpha\sum_{i=1}^{n}\frac{\left(1+\frac{\lambda}{x_i}\right)^{\alpha-1}}{x_i}-\alpha\left[\frac{n}{\sum_{i=1}^{n}\ln\left[1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right]}+1\right]\sum_{i=1}^{n}\frac{\left(1+\frac{\lambda}{x_i}\right)^{\alpha-1}e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}}{x_i\left[1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right]}=0 \quad (32)$$

These nonlinear equations cannot be analytically solved, but the statistical software like R program (Team (2015)) can be used to solve them numerically using iterative techniques.

The asymptotic variance-covariance matrix of the ML estimators for the two parameters α and λ is the inverse of the observed Fisher information matrix as follows

$$\hat{F}=\begin{bmatrix}-\frac{\partial^2\ln L}{\partial\alpha^2} & -\frac{\partial^2\ln L}{\partial\alpha\partial\lambda} & -\frac{\partial^2\ln L}{\partial\alpha\partial\beta}\\ -\frac{\partial^2\ln L}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ln L}{\partial\lambda^2} & -\frac{\partial^2\ln L}{\partial\lambda\partial\beta}\\ -\frac{\partial^2\ln L}{\partial\beta\partial\alpha} & -\frac{\partial^2\ln L}{\partial\beta\partial\lambda} & -\frac{\partial^2\ln L}{\partial\beta^2}\end{bmatrix}^{-1}=\begin{bmatrix}\hat{\sigma}_{\alpha}^2 & \hat{\sigma}_{\alpha,\lambda} & \hat{\sigma}_{\alpha,\beta}\\ \hat{\sigma}_{\lambda,\alpha} & \hat{\sigma}_{\lambda}^2 & \hat{\sigma}_{\lambda,\beta}\\ \hat{\sigma}_{\beta,\alpha} & \hat{\sigma}_{\beta,\lambda} & \hat{\sigma}_{\beta}^2\end{bmatrix}$$

The elements of the sample Fisher information matrix are obtained by deriving the second derivatives of the log-likelihood function (29) and evaluating them at the MLEs ((Cohen 1965)). The second derivatives of the log-likelihood function are

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \sum_{i=1}^{n}\left(1+\frac{\lambda}{x_i}\right)^{\alpha} \ln^2\left(1+\frac{\lambda}{x_i}\right) + (\beta-1) \times \sum_{i=1}^{n}\left(\ln\left(+\frac{\lambda}{x_i}\right)\right)\left\{\frac{\left\{\begin{array}{c}\left(1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)\left(1+\frac{\lambda}{x_i}\right)^{\alpha} \\ \times\left(-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}\ln\left(1+\frac{\lambda}{x_i}\right)e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right) \\ +e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\left(1+\frac{\lambda}{x_i}\right)^{\alpha}\ln\left(1+\frac{\lambda}{x_i}\right)\end{array}\right\} -\left(\left(1+\frac{\lambda}{x_i}\right)^{\alpha}e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)\left(e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\left(1+\frac{\lambda}{x_i}\right)^{\alpha}\ln\left(1+\frac{\lambda}{x_i}\right)\right)}{\left(1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)^2}\right\}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n}{\beta^2}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -\frac{n}{\lambda^2} + (\alpha-1)\sum_{i=1}^{n}\frac{-x^{-2}}{\left(1+\frac{\lambda}{x_i}\right)^2} - \alpha\sum_{i=1}^{n}x^{-2}(\alpha-1)\left(1+\frac{\lambda}{x_i}\right)^{\alpha-2} + (\beta-1)\sum_{i=1}^{n}\frac{\alpha}{x_i}\left\{\frac{\left\{\begin{array}{c}\left(1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)\left(1+\frac{\lambda}{x_i}\right)^{2\alpha-2}\left(-\frac{\alpha}{x_i}e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right) \\ +\frac{\alpha-1}{x_i}\left(1+\frac{\lambda}{x_i}\right)^{\alpha-2}e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\end{array}\right\} - \left\{\frac{\alpha}{x_i}\left(e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\left(1+\frac{\lambda}{x_i}\right)^{\alpha-1}\right)^2\right\}}{\left(1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)^2}\right\}$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda}=\sum_{i=1}^{n}\frac{-x_i^{-1}}{1+\frac{\lambda}{x_i}}-\sum_{i=1}^{n}\left\{\left(1+\frac{\lambda}{x_i}\right)^{\alpha}\left(\frac{-x_i^{-1}}{1+\frac{\lambda}{x_i}}\right)+\frac{\alpha}{x_i}\left(1+\frac{\lambda}{x_i}\right)^{\alpha}\ln\left(1+\frac{\lambda}{x_i}\right)\right\}$$
$$+(\beta-1)$$
$$\times\sum_{i=1}^{n}\left\{\begin{array}{c}\left\{\begin{array}{c}\left(1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)\left\{\left(1+\frac{\lambda}{x_i}\right)^{\alpha}\left(e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)\left(\frac{x_i^{-1}}{1+\frac{\lambda}{x_i}}\right)\right\}\\ +\left\{\left(1+\frac{\lambda}{x_i}\right)^{\alpha}\ln\left(1+\frac{\lambda}{x_i}\right)\left(-\frac{\alpha}{x_i}e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\left(1+\frac{\lambda}{x_i}\right)^{\alpha-1}\right)\right\}\\ +\left\{\left(e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)\ln\left(1+\frac{\lambda}{x_i}\right)\left(\frac{\alpha}{x_i}\left(1+\frac{\lambda}{x_i}\right)^{\alpha-1}\right)\right\}\end{array}\right\}\\ \frac{-\left\{\left(1+\frac{\lambda}{x_i}\right)^{\alpha}\left(e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)\ln\left(1+\frac{\lambda}{x_i}\right)\right\}\left\{\frac{\alpha}{x_i}e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\left(1+\frac{\lambda}{x_i}\right)^{\alpha-1}\right\}}{\left(1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)^{2}}\end{array}\right\}$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\beta}=\sum_{i=1}^{n}\frac{e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\ln\left(1+\frac{\lambda}{x_i}\right)\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}{1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}}$$

$$\frac{\partial^2 \ln L}{\partial\lambda\partial\beta}=\sum_{i=1}^{n}\frac{\alpha}{x_i}\left(\frac{\left\{\begin{array}{c}\left(1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)\left(1+\frac{\lambda}{x_i}\right)^{2\alpha-2}\left(-\frac{\alpha}{x_i}e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)\\ +\frac{\alpha-1}{x_i}\left(1+\frac{\lambda}{x_i}\right)^{\alpha-2}e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\end{array}\right\}-\left\{\frac{\alpha}{x_i}\left(e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\left(1+\frac{\lambda}{x_i}\right)^{\alpha-1}\right)^{2}\right\}}{\left(1-e^{1-\left(1+\frac{\lambda}{x_i}\right)^{\alpha}}\right)^{2}}\right)$$

We can employ the asymptotic normality of the MLE to compute the approximate confidence intervals for the parameters α, λ and β, as follow

$$\hat{\alpha}_{ML}\pm z_{\tau/2}\sqrt{\hat{\sigma}_{\alpha}^{2}}\ ,\qquad \hat{\lambda}_{ML}\pm z_{\tau/2}\sqrt{\hat{\sigma}_{\lambda}^{2}}\quad and\quad \hat{\beta}_{ML}\pm z_{\tau/2}\sqrt{\hat{\sigma}_{\beta}^{2}}\qquad (33)$$

where $z_{\tau/2}$ is an upper $(\tau/2)100\%$ of the standard normal distribution.

5. Real Data Illustration

In this section under EINH we will present two kinds of real data and apply the statistical estimations of unknown parameters to compare with other distributions and see the flexibility among them.

The first data set represents the remission times (in months) of a random sample of 128 bladder cancer patients see Lee and Wang (2003). The data are: 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63.

The second real data consists of the number of successive failure for the air conditioning system reported of each memberin a fleet of 13 Boeing 720 jet airplanes. The pooled data with 214 observations was considered by Proschan (1963), Kus (2007) and many others. The data are: 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.

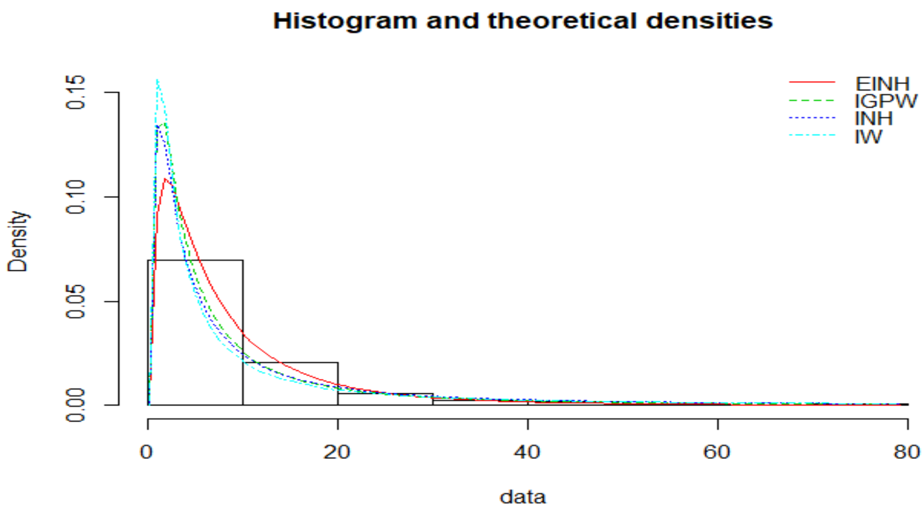
We fitted the above-mentioned data to the exponentiated inverse Nadarajah-Haghighi (EINH), inverse generalized power Weibull (IGPW) by Selim (2019), inverse Nadarajah-Haghighi (INH) by Tahir et al. (2018), inverse Weibull (IW), inverse exponential (IE) and inverse Rayleigh (IR) distributions. The MLEs and their standard errors for EINH, INH, IW and IE distributions, along with statistics criteria like, -Maximized Loglikelihood $(-L)$, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion $(CAIC)$, Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion $(HQIC)$ are computed for our data and displayed in Tables 1 , 2. The best model is the one that acquires the lowest values for the information criteria. Hence, it is clear from the numerical results in Tables 1, 2 that the EINH model provides a better fit than the other competitive models. The Figure 5,6 display the graphical comparison of EINH, IGPW, INH and IW distributions. This figure also illustrate that EINH distribution provides the best fit to our data set as compared to the other models. Therefore, the EINH model can be used as a possible alternative to the well- known inverse exponential and inverse Weibull models.

Table 1. The estimates and the standard errors (in parentheses) and goodness-of-fit statistics for bladder cancer patients data

Model	α	λ	β	K-S	W*	A*	-L	AIC	CAIC	BIC	HQIC
EINH	**0.2993 (0.032)**	**400.3205 (336.39)**	**8.5438 (3.851)**	**0.0563**	**0.0822**	**0.5647**	**413.182**	**832.362**	**832.556**	**840.918**	**835.839**
IGPW	0.4435 (0.048)	15.7165 (4.8326)	1.2110 (0.102)	0.1364	0.3368	2.1713	426.910	859.819	860.013	868.375	863.296
INH	0.5064 (0.048)	10.5947 (2.322)	-	0.1636	0.3565	2.2844	431.059	866.118	866.214	871.822	868.436
IW	2.4311 (0.219)	0.7521 (0.0424)	-	0.1408	0.7443	4.5464	444.001	892.002	892.098	897.706	894.319
IE	-	2.4847 (0.220)	-	0.2311	1.1139	6.6074	460.382	922.765	922.796	925.617	923.923
IR	-	0.6174 (0.0545)	-	0.7502	2.3754	13.2264	774.342	1550.68	1550.72	1553.54	1551.84

Table 2. The estimates and the standard errors (in parentheses) and goodness-of-fit statistics for the number of successive failure for the air conditioning system

Model	α	λ	β	K-S	W*	A*	-L	AIC	CAIC	BIC	HQIC
EINH	**0.3898 (0.032)**	**300.1352 (116.78)**	**2.0089 (0.397)**	**0.0564**	**0.0766**	**0.6104**	**981.90**	**1969.80**	**1969.93**	**1979.36**	**1973.67**
IGPW	0.3793 (0.0414)	299.9249 (149.459)	1.2219 (0.087)	0.0565	0.1291	0.9764	986.22	1978.45	1978.58	1988.01	1982.32
INH	0.4809 (0.0428)	89.6721 (18.894)	-	0.0831	0.1849	1.3379	991.19	1986.38	1986.45	1992.75	1988.97
IW	10.9428 (1.3332)	0.7478 (0.0385)	-	0.1010	0.4605	3.0918	1003.86	2011.73	2018.10	2018.10	2014.31
IE	-	19.3021 (1.4427)	-	0.1924	0.7146	4.6424	1023.60	2049.20	2049.22	2052.38	2050.49
IR	-	61.7666 (4.6167)	-	0.6120	1.7559	10.6119	1378.56	2759.13	2759.13	2762.32	2760.42


Histogram and theoretical densities
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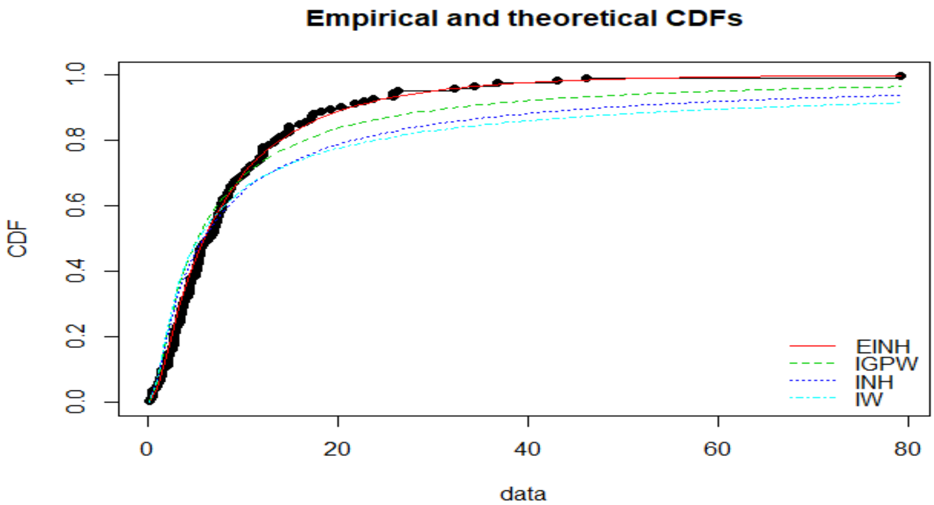
0.10
0.15
0
20
40
60
80
data
EINH
IGPW
INH
IW


Empirical and theoretical CDFs
CDF
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0.6
0.8
1.0
0
20
40
60
80
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EINH
IGPW
INH
IW

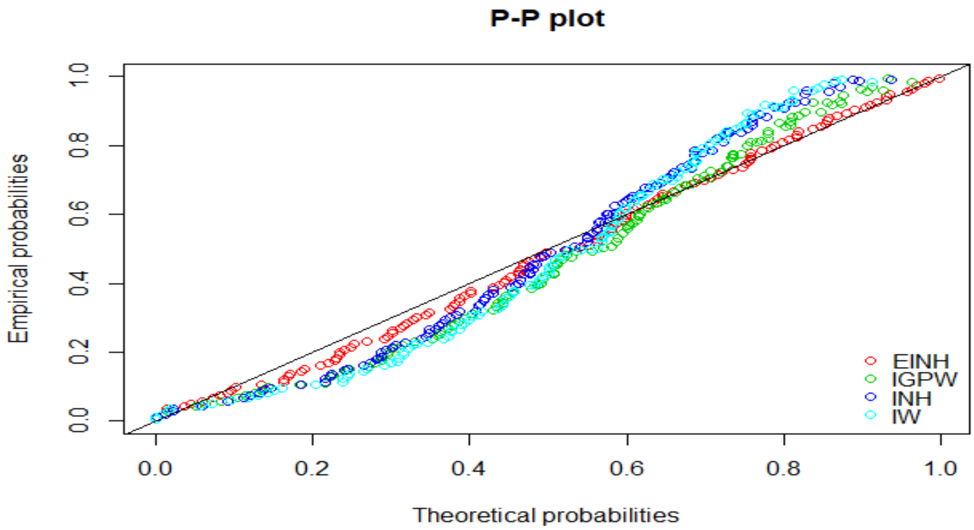
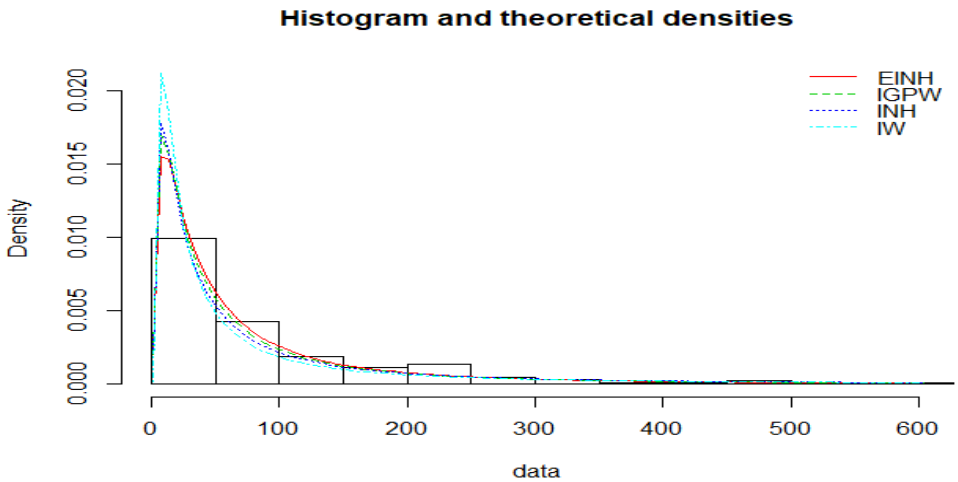


Fig. 5:Histogram and estimated densities (Upper Panel); Theoretical and estimated CDFs (Middle Panel); P-P plots (Bottom Panel) for bladder cancer patients data


Histogram and theoretical densities
Density
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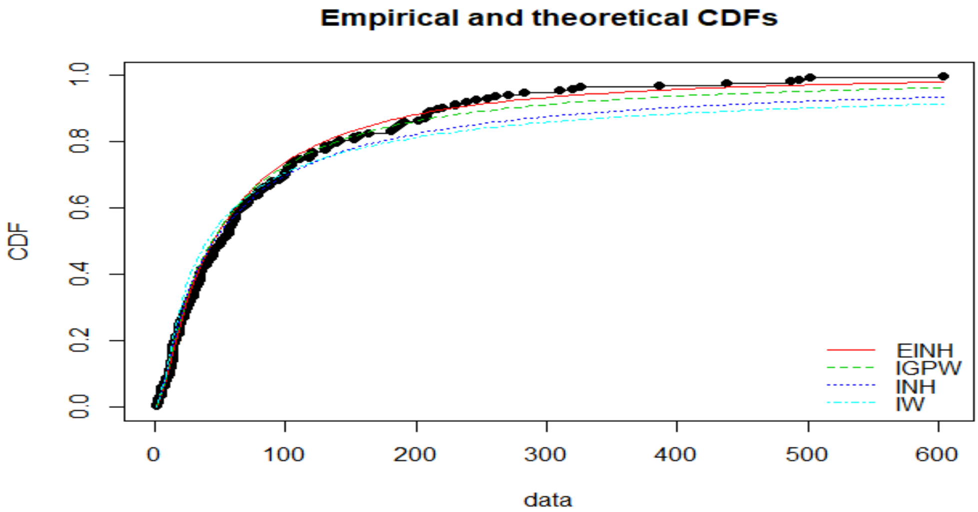
0.015
0.020
0
100
200
300
400
500
600
data
EINH
IGPW
INH
IW


Empirical and theoretical CDFs
CDF
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0.2
0.4
0.6
0.8
1.0
0
100
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300
400
500
600
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EINH
IGPW
INH
IW

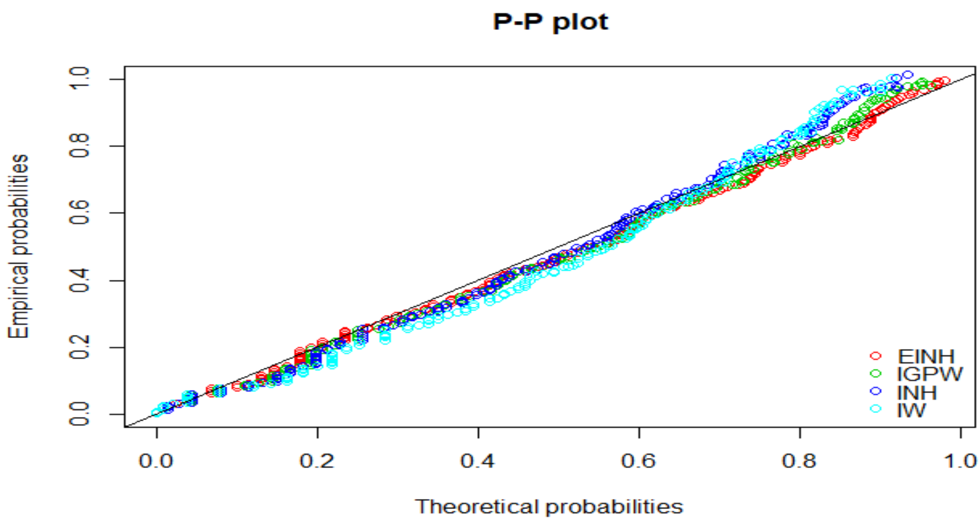

Fig. 6:Histogram and estimated densities (Upper Panel); Theoretical and estimated CDFs (Middle Panel); P-P plots (Bottom Panel) for the number of successive failure for the air conditioning system

6. Conclusion

In this article, we define a new model called the exponentiated inverse Nadarajah-Haghighi distribution (EINH). This new model is a generalization of the inverse Nadarajah-Haghighgi distribution, which is introduced by Taher et al (2018). Some of its mathematical properties are derived. Order statistics functions are also derived. The shapes of pdf, cdf and hazard function are displayed to show the flexibility of the new model. The model parameters are estimated by the maximum likelihood method. The two kinds of real data are used as an application of EINH. The results of the application are shown that the EINH is more flexible than the related models.

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