Time dependent of an $M/M/c/N$ with discouragement arrivals and retention of impatient customers

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Abstract:
In this paper, the matrix technique is employed to attain transient solution of a finite capacity multi-servers Markovian queue having discouraged arrivals, retention and reneging of reneged customers. We consider the cases where both service times and the inter-arrival follow the exponential distribution. The time dependent solution of the system is expressed by using the eigenvalues of a symmetric tridiagonal matrix. As a reneged customer can be preserved in a lot of cases by utilizing definite convincing mechanisms to keep on queue for completion of service, it follows that a reneged customer can keep hold of the queuing system with chance q. Otherwise, customer may leave the queue without getting a service with probability p ($p+q=1$). Laplace transforms of governing equations system are expressed in matrix forms. Moreover, employing the characteristics of symmetric tridiagonal matrices, and the steady state probabilities are gained. Finally, some queuing models are derived as special cases of this system.

Keyword: Reneging, retention of reneged customers, transient solution, Discouraged arrivals.

1- Introduction

The study of a finite waiting space queuing systems is very demanding and important since these systems represent a commonly observed type of systems in queuing theory. Therefore, many researchers have studied queuing systems with a finite waiting capacity. For example, Sumeet and Rakesh [1] studied a feedback Markovian queue having retention of reneged customers in steady–state. Chandrika [2] investigated transient queue system with dependent servers and controllable arrivals. Kumar and Sharma[3] successfully obtained the steady state solution of queuing model with discouraged arrivals, reneging and retention of reneged customers. Also, the cost and profit function are acquired. Thiagarajan and Premalatha [4] examined single–server infinite waiting space simple queue with balking, retention of reneged customers and feedback.

The transient solution of finite capacity single-server queue is studied by Al-Seedy et al. [5] via employing the matrix techniques. Furthermore, the case of Transient behavior of an $M/M/1/N$ queue with multiple working breakdowns is considered by Yang and Wu[6]. Jain and Bura [7] examined the effect of varying catastrophic intensity with restoration by utilizing a simple finite capacity Markovian queue with capacity N. The time dependent analysis of a single–server queue having constant-size batch arrivals is analyzed by Oduol and Ardil [8] whereas the analysis of transient solution of a two-heterogeneous servers queue having impatient concepts is carried out by S. Ammar [9].Kumar and Sharma [10]studied the optimization of a simple queue with feedback and retention of reneged customers. Also, the optimization problem of an $M/M/1/N$ queuing system possessing impatient customers, variable input rates, and different service rates is solved by Pan [11]. Majewska [12] considered Analysis and optimization of queuing system with impatient customers. Tian et al. [13] considered the optimal balking behavior of customers in an M/G/1 queuing system having a removable server under N-policy. Finite capacity multi-server queuing systems investigated numerically by Kumar and Sharma[14]. Moreover, Kumar [15] considered the behaviors of multi-server Markovian queuing models including balking and catastrophes via utilizing the probability generating function scheme.

In this paper, time dependent and steady state results are obtained for the multi- servers queue having discouraged arrivals, reneging, and retention of reneged customers using matrix technique.

2- Model description

Assume that we have an $M/M/c/N$ queue with reneging, retention of reneged customers and discouraged arrivals. Capacity of this system is considered finite, say N. Customers are supposed to arrive at service station in a one by one fashion according to Poisson stream with arrival rate λ_n . The arrival rate is dependent on the customers number that is present in the system at time i.e. $\lambda_n = \lambda\gamma_n, \gamma_n = \frac{1}{(n-c+1)+1}, n = c, c+1, ..., N-1$ but $\gamma_n = 1, n = 0,1,...,c-1$ and $\gamma_N = 0$.

There is a multi-servers, denoted by c servers, which provides service to all arriving customers. Service times are independent and identical exponential distribution random variables with queue parameter μ . Note that queue discipline is of first-in , first out(FIFO) type. The customer in queue, for regular arrival case, may be impatient when the required service is not accessible for a considerable time T. This long time T is a random variable having the following probability distribution: $f(t) = \alpha e^{-\alpha t}$, $t \geq 0, \alpha > 0$, where α is the rate of time T. Also, T is reneging time of a particular customer after which customer either choose to leave the queue with probability p ($p + q = 1$) or never to return with complementary probability.

3- Governing Equations

Assume that $p_n(t)$ is the probability that the system size equals n $(n = 0,1,...,N)$ at time t, .Also, assume that the empty system is starting with i customers (the system size at $t = 0$).

The system is governed by the following set of differential-difference equations:

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad (1)$$

$$P_n'(t) = -(\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t) + \lambda_{n-1} P_{n-1}(t), 1 \le n \le c-1 \quad (2)$$

$$P_n'(t) = -(\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t) + \lambda_{n-1} P_{n-1}(t), c \le n \le N-1 \quad (3)$$

$$P_N'(t) = \lambda_{N-1} P_{N-1}(t) - \mu_N P_N(t), n = N \quad (4)$$

Where:

$$\lambda_n = \begin{cases} \lambda, 0 \le n \le c-1 \\ \dfrac{\lambda}{(n-c)+2}, c \le n \le N-1 \end{cases}, \quad \mu_n = \begin{cases} n\mu, 1 \le n \le c-1 \\ c\mu + (n-c)\alpha p, c \le n \le N \end{cases}$$

The Laplace transform of the differential difference equations (1)-(4) are expressed as

$$\left(s + \lambda_0\right) P^*_0(s) = \mu_1 P^*_1(s) + \delta_{i0}, \ n = 0 \quad (5)$$

$$(s + \lambda_n + \mu_n) P^*_n(s) = \lambda_{n-1} P^*_{n-1}(s) + \mu_{n+1} P^*_{n+1}(s) + \delta_{in}, \ 1 \le n \le c-1 \quad (6)$$

$$(s + \lambda_n + \mu_n) P^*_n(s) = \lambda_{n-1} P^*_{n-1}(s) + \mu_{n+1} P^*_{n+1}(s) + \delta_{in}, \ c \le n \le N-1 \quad (7)$$

$$(s + \mu_N) P^*_N(s) = \lambda_{N-1} P^*_{N-1}(s) + P_N(0), \ n = N \quad (8)$$

The equations (5)-(8) can be given in the following matrix form: $A(s)P(s) = P(0)$,

where

$$P(s) = \left(P^*_0(s), P^*_1(s), ..., P^*_N(s)\right)^T, \quad P(0) = \left(\delta_{i0}, \delta_{i1}, ..., \delta_{iN}\right)^T,$$

and $p_n(0) = \delta_{in}$ (δ_{in} is the usual Kronecker delta), and the Laplace transform of the probability $P_n(t)$ is $P^*_n(s) = \int_0^\infty e^{-st} P_n(t)\,dt$.

The matrix $A(s)$ with order $N+1$ is expressed as follows

$$\begin{bmatrix} s+\lambda_0 & -\mu_1 & 0 & 0\ldots & 0 & 0 \\ -\lambda_0 & s+\lambda_1+\mu_1 & -\mu_2 & 0\ldots & 0 & 0 \\ 0 & -\lambda_1 & s+\lambda_2+\mu_2 & -\mu_3\ldots & 0 & 0 \\ 0 & 0 & -\lambda_2 & s+\lambda_3+\mu_3\ldots & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0\ldots & s+\lambda_{N-1}+\mu_{N-1} & -\mu_N \\ 0 & 0 & 0 & 0\ldots & -\lambda_{N-1} & s+\mu_N \end{bmatrix},$$

As $A(s)$ is similar to a symmetric tri-diagonal matrix, then its eigenvalues are real.

Employing some basic row and column transformations on $|A(s)|$, $|A(s)| = s\,|M(s)|$ is hold, where $M(s)$ is an order N symmetric tri-diagonal matrix with negative off diagonal elements. Matrix $M(s)$ is written as

$$\begin{bmatrix} s+\lambda_0+\mu_1 & -\sqrt{\lambda_1\mu_1} & 0 & 0... & 0 & 0 \\ -\sqrt{\lambda_1\mu_1} & s+\lambda_1+\mu_2 & -\sqrt{\lambda_2\mu_2} & 0... & 0 & 0 \\ 0 & -\sqrt{\lambda_2\mu_2} & s+\lambda_2+\mu_3 & -\sqrt{\lambda_3\mu_3}... & 0 & 0 \\ 0 & 0 & -\sqrt{\lambda_3\mu_3} & s+\lambda_3+\mu_4... & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0... & s+\lambda_{N-2}+\mu_{N-1} & -\sqrt{\lambda_{N-1}\mu_{N-1}} \\ 0 & 0 & 0 & 0... & -\sqrt{\lambda_{N-1}\mu_{N-1}} & s+\lambda_{N-1}+\mu_N \end{bmatrix}$$

Notice that the solutions of equation $|M(s)|=0$ are the negative eigenvalues of the matrix $M(s)$. Since the matrix M is positive definite($-\mu_i$.$-\lambda_i$ >0), real and symmetric, the eigenvalues of M are distinct, real and positive. Moreover, the roots of the polynomial $|A(s)|$ are distinct, real and negative. Denote these roots by $s_0(=0), s_1,...,s_N$, then we get:

$$|A(s)| = s(s-s_1)...(s-s_N) \text{ and}$$

$$p^*_n(s) = \frac{|A_{n+1}(s)|}{s(s-s_1)...(s-s_N)}, \tag{9}$$

where $|A_{n+1}(s)|$ is the determinants of the matrix attained via replacing the $(n+1)^{\text{th}}$ column of $A(s)$ with the column vector $P(0)$. Assume $T_r(s)$ and $B_r(s)$ are the determinants of the top left and bottom right $r \times r$ submatrices extracted from the matrix $A(s)$. Therefore, the following recurrence relations are satisfied by determinants $T_r(s)$ and $B_r(s)$

$$B_r(s) - a_{N-r+1}\, B_{r-1}(s) + b_{N-r+1}\, c_{N-r+1}\, B_{r-2}(s) = 0,\ r = 2,3,\dots,N$$

$$T_r(s) - a_{r-1} T_{r-1}(s) + b_{r-2}\, c_{r-2}\, T_{r-2}(s) = 0,\ r = 2,3,\dots,N \quad ,$$

For the next initial conditions

$$B_0(s) = 1, \quad B_1(s) = a_N\ ,\ T_0(s) = 1, \quad T_1(s) = a_0.$$

where:

$$b_i = \begin{cases} (i+1)\mu,\ i = 0,1,\dots,c-1 \\ [c\mu + (i-c+1)\alpha p],\ i = c,\dots,N \end{cases},$$

$$c_i = \begin{cases} \lambda,\ i = 0,1,\dots,c-2 \\ \dfrac{\lambda}{i-c+2},\ i = c-1,\dots,N-1 \end{cases}, \text{ and}$$

$$a_i = \begin{cases} s + c_0,\ i = 0 \\ s + c_i + b_{i-1},\ i = 1,\dots,N-1 \\ s + b_{N-1}, i = N \end{cases}.$$

Now, the expression of $|A_{n+1}(s)|$ in terms of $T_r(s)$ and $B_r(s)$ is to be obtained.

For $n \ge i$, the value of n belongs to one of the following intervals:

(i) $0 \le n \le c$

(ii) $c < n \le N$

Thus, at $0 \le n \le c$, we have

$$|A_{n+1}(s)| = T_i(s)\lambda^{n-i}\, B_{N-n}(s),\ 0 \le i \le n. \qquad (10)$$

Moreover, for $c < n \le N$,

$$|A_{n+1}(s)| = \begin{cases} T_i(s)\lambda^{c-i} \displaystyle\prod_{k=c}^{n-1} \lambda_k\, B_{N-n}(s), & 0 \le i < c \\ T_i(s) \displaystyle\prod_{k=i}^{n-1}\, B_{N-n}(s), & c \le i < N. \end{cases} \qquad (11)$$

Hence for $n < i$, n can be assigned to one of the following intervals:

(i) $0 \le n < c$

(ii) $c \le n \le N$

The form of $|A_{n+1}(s)|$ is given by

for $0 \le n < c$,

$$|A_{n+1}(s)| = \begin{cases} T_n(s)\dfrac{i!}{n!}\mu^{i-n} B_{N-i}(s), & n < i < c \\ T_n(s)\dfrac{c!}{n!}\mu^{c-n}\prod_{k=c}^{i-1}\mu_k \; B_{N-i}(s), & c \le i \le N, \end{cases} \tag{12}$$

and for $c \le n \le N$,

$$|A_{n+1}(s)| = T_n(s)\prod_{k=n}^{i-1}\mu_k \; B_{N-i}(s), n < i \le N. \tag{13}$$

From Equation (9) and using partial fraction, we derive that

$$p^*_n(s) = \frac{p_n}{s} + \sum_{k=1}^{N}\frac{d_{n,k}}{s - s_k}, \tag{14}$$

where

$$p_n = \lim_{s\to 0} s\, p^*_n(s), \qquad d_{n,k} = \lim_{s\to s_k}(s - s_k)\, p^*_n(s).$$

By inverting the Laplace transforms of equation (14), we acquire the following expression in the time domain

$$p_n(t) = p_n + \sum_{k=1}^{N} d_{n,k}\, e^{s_k t} \tag{15}$$

4-Steady state probabilities

Define p_n as the equilibrium probability of n customers exist in the system. Then from equation (7) we have:

$$p_n = \lim_{s\to 0} s\, p^*_n(s) = \frac{|A_{n+1}(0)|}{\prod_{i=0}^{N}(-s_i)}, n = 0,1,\ldots N. \tag{16}$$

From the aforementioned formula of $T_r(s), B_r(s)$ and the difference equations, we can obtain

$$T_n(0)=\frac{\lambda^n}{n!},\; B_{N-n}(0)=\prod_{K=1}^{N-n}\mu_k \text{ and } T_0(0)=1.$$

For $n=0$, $p_0=\frac{|A_1(0)|}{\prod_{i=0}^{N}(-s_i)} \Rightarrow p_n=\frac{|A_{n+1}(0)|}{|A_1(0)|}p_0, n=0,1,...N$, (17)

$$p_n=\frac{|A_{n+1}(0)|}{|A_1(0)|}p_0=\frac{T_n(0)b_n b_{n+1}...b_{i-1}B_{N-i}(0)}{T_0(0)b_0 b_1..b_{i-1}B_{N-i}(0)}p_0,\; n<i\;, \qquad (18)$$

$$p_n=\begin{cases}\frac{\lambda^n}{n!\mu^n}p_0; 1\le n\le c\\ \frac{\lambda^n}{c!\mu^c(n-c+1)!}\frac{p_0}{\prod_{k=c+1}^{n}[c\mu+(k-c)\alpha p]}; c+1\le n\le N.\end{cases} \qquad (19)$$

Using (10), (12), (13) and (16) it is concluded that the formula in (19) is independent of the initial size of system i .The result in (19) agrees with that in Thiagarajanand Premalatha [4].

Placing the expressions of p_n, $n=0,1,...,N$, in normalized equation, the expression for p_0 are determined as follows

$$p_0=\left[1+\sum_{k=1}^{c}\frac{\lambda^k}{k!\;\mu^k}+\frac{1}{c!\mu^c}\sum_{k=c+1}^{N}\frac{\lambda^k}{(k-c+1)!}\prod_{j=c+1}^{k}\frac{1}{c\mu+(j-c)\alpha p}\right]^{-1}. \qquad (20)$$

5- Expected queue length for customers

Let $Q(t)$ be the random variable denoting queue size where $E[Q(t)]$ refers to its expected value. Then,

$$L(t)=E[Q(t)]=\sum_{n=1}^{N}n\,p_n(t)=\sum_{n=1}^{N}n\,p_n+\sum_{n=1}^{N}n\sum_{k=1}^{N}d_{n,k}\,e^{s_k t}. \qquad (21)$$

6- Special Cases

1- In the above described model taking $c = 1, 0 < p < 1,$ and $t \to \infty$, we shall reach to an $M/M/1/N$ queueing system having reneging and retention of reneged customers as studied by Kumaret al. [16].

2- When $t \to \infty$ the queueing system transformed into an $M/M/c/N$ queueing model in steady state with discouraged arrivals, reneging and retention of reneged customers the result agrees with those presented by Kumarand Sharma [14].

3-Setting $p = 1(q = 0, \alpha = 0), c = 1,$ and $t \to \infty$ the model will be approached which would be the system $M/M/1/N$ without any concepts.

7- Numerical illustrations of the model

In this part, some numerical values for a virtual model are considered where the transient expected value of queue size as function of the time t is computed. Furthermore, the transient expected value of queue length for t values is illustrated in figure 1. From the plotted figure it is obvious that the model gets to the steady state at point $t = 2$. The transient expected value of queue length for $t \geq 2$ equivalents with the steady state value of it. In figure 2 noted that the transient expected value of queue size of the model increases with increasing the λ values and decreases with value of μ.In figure 3 empty transient probability is decreasing with increasing value of λ.

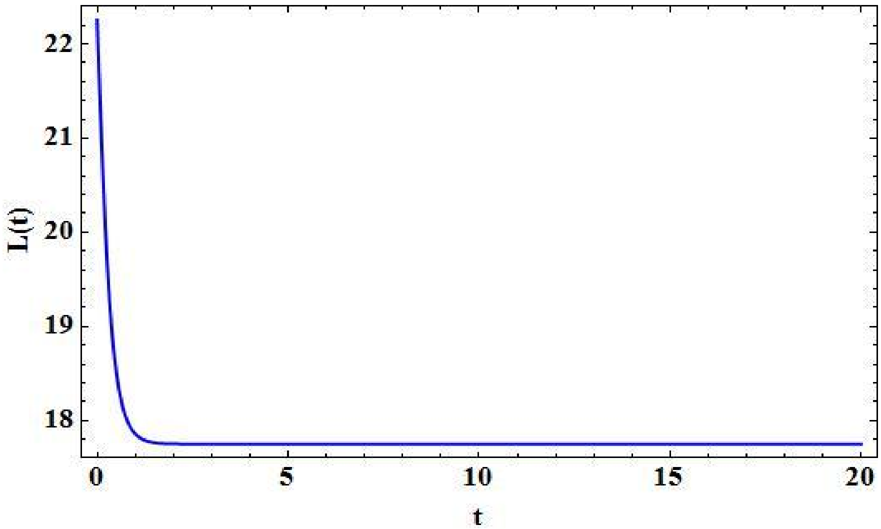

Figure 1. The expected value of queue length for different times are plotted for the case

$\alpha = 0.2, c = 3, p = 0.4, i = 5, \lambda = 2, \mu = 4, N = 15.$

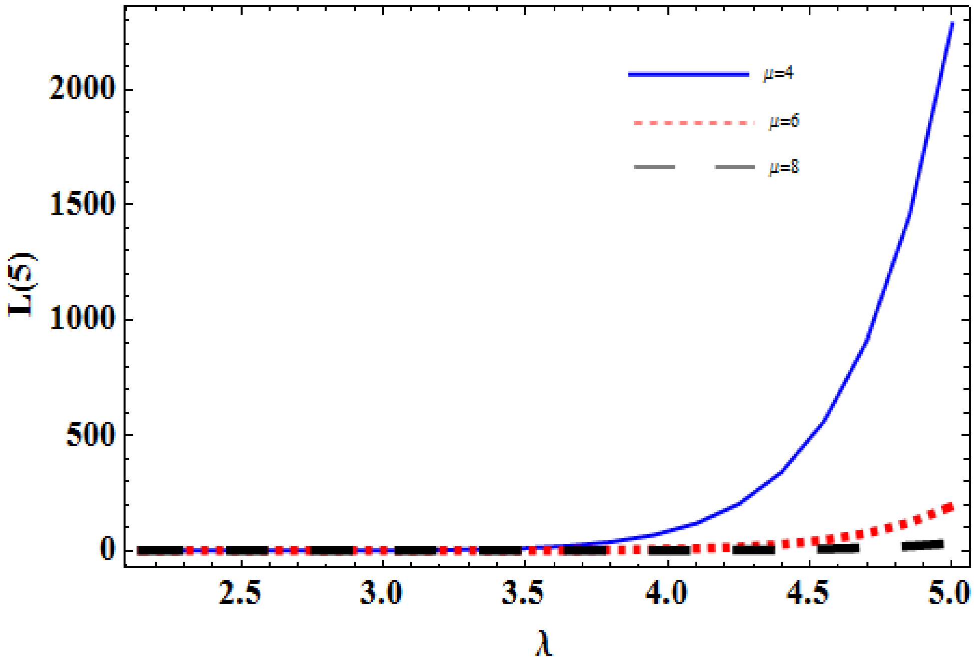

Figure 2. Variation of the expected value of queue length for different values of arrival rate and service rate at time point equal 5 for the values $\alpha = 0.2$, $p = 0.4$, $i = 5$,, $t = 5$ and $N = 15$.

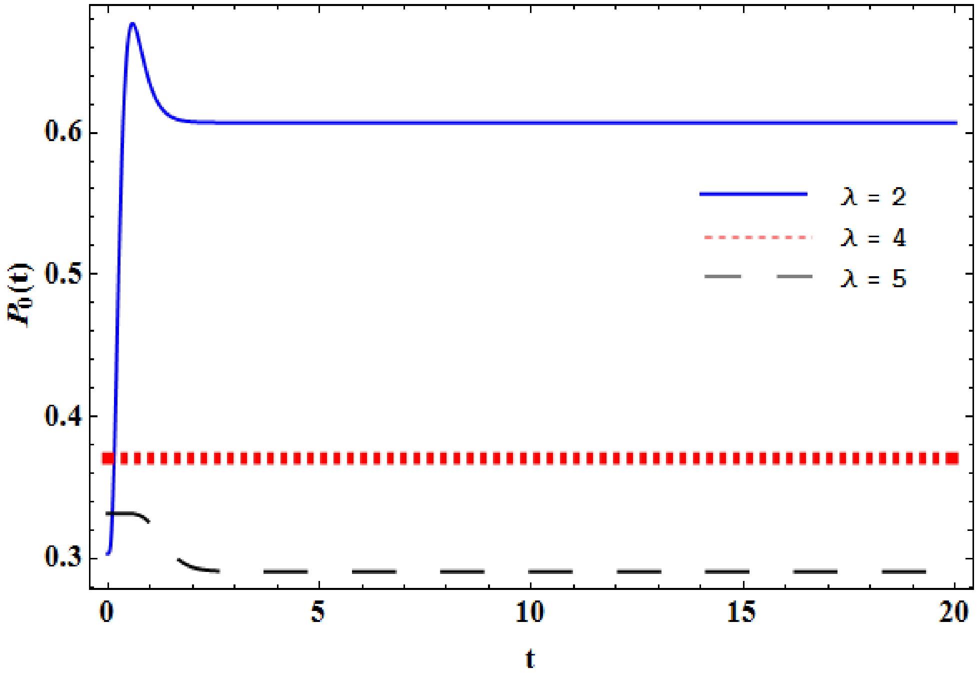

Figure 3. $P_0(t)$ versus t for different values of λ

8- Conclusion

This work examines a multi- servers queuing system having retention of reneged customers and discouraged arrivals. The time dependent solution and steady state are obtained in explicit form by using a computable matrix technique. Special cases of queueing models are derived for this model. From the plotted figure it is found that the system attains the value of steady state at time $t = 2$, the transient expected value of queue size of the model decreases with increasing the values of λ. Also, The empty transient probability $P_0(t)$ is decreasing with increasing value of λ.

Appendix:

From the identities of tridiagonal determinants Muir and Metzler [17] is:

$$\begin{vmatrix} A+a_0 & a_0 & & & & \\ b_1 & A+a_1+b_1 & a_1 & & & \\ & . & & & & \\ & & . & & & \\ & & & . & & \\ & & & & A+a_{n-1}+b_{n-1} & a_{n-1} \\ & & & & b_n & A+b_n \end{vmatrix}$$

$$= A \times \begin{vmatrix} g_1 & b_1 & & & & & \\ a_1 & g_2 & b_2 & & & & \\ & & . & & & & \\ & & & . & & & \\ & & & & . & & \\ & & & & a_{n-2} & g_{n-1} & b_{n-1} \\ & & & & & a_{n-1} & g_n \end{vmatrix}$$

where $g_i = A + a_{i-1} + b_i,\ i = 1, 2, ..., n.$

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