Comparison of *Direct* L-moments, L-moments and ML Estimation Methods for Weibull Distribution with Type-I Censoring

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ABSTRACT

This paper presents a comparison of three different methods, *Direct* L-moments, L-moments via partial probability-weighted moments (PPWM) and maximum likelihood (ML) methods, respectively, to estimate the two parameters of Weibull distribution with Type-I censored data. These methods are compared in terms of estimate of the unknown parameters, relative bias and root of mean square error (RMSE) using Monte Carlo simulation to select the best method. Also, a real data set is considered to achieve the results.

Keywords: *Censored Data; Estimation; Direct L-moments; L-moments; maximum likelihood; Weibull Distribution.*

1- Introduction

Weibull distribution is widely used in reliability and survival analysis. It has been introduction by Wallodi Weibull 1951. This article is concerned with the two parameters Weibull distribution. The probability density function (pdf) of Weibull distribution is:

$$f(x;a,b) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^{b}}, \qquad 0 < x < \infty, \qquad a, b > 0 \tag{1.1}$$

and the cumulative distribution function (cdf) is:

$$F(x) = 1 - e^{-\left(\frac{x}{a}\right)^{b}} \tag{1.2}$$

The quantile function can be obtained as follows:

$$x(u) = a\left[-\log(1-u)\right]^{\frac{1}{b}} \tag{1.3}$$

This paper is concerned with estimating the two parameters *a* and *b* of Weibull distribution using three different methods, *Direct* L-moments with L-moments via partial probability-weighted moments (PPWM) method and maximum likelihood (ML) method, respectively with Type-I censored data.

This article is organized as follows; L-moments via PPWM for Censored Data for Weibull distribution are introduced in Section 2.

Direct L-moments for censored data for Weibull distribution is presented in Section 3. ML method for censored data for Weibull distribution is presented in Section 4. in Section 5 an application of a real data set is presented. Simulation study and concluding remarks are presented in section 6 and 7 respectively.

2- L-moments via PPWM for Censored Data for Weibull distribution

L-moments was first introduced by Sillitto (1951) and formally defined as expectations of certain linear combinations of order statistics by Hosking (1990). He has defined L-moments to summaries theoretical distribution and observed samples. Hosking (1995) defined two variants of L-moments definition, which used with right-censored data. Similarly, Zafirakou-Koulouris *et al.* (1998) extended the applicability of L-moments to left-censored data of L-moments. The method of L-moments was discussed from various earlier studies by Hosking (1990) who provided guidelines for the practical use of L-moments. Hosking (1990, 1992, 1994, 2006), Hosking and Wallis (1995), Sillitto (1969), Elamir and Seheult (2001, 2004), Jones (2004), Asquith (2007), Abdul-Moniem (2007), Karvanen (2006), Bílková (2014) and Dutang (2016) considered various theoretical aspects and applications of L-moments for complete data.

For any distribution the r^{th} L-moments is related to the r^{th} PPWM, see Hosking (1990), where:

$$\lambda_{r+1} = \sum_{k}^{r} (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} \beta_k, \qquad r = 0, 1, 2, \ldots\ldots \tag{2.1}$$

from which the first four L-moments in terms of PPWM are:

$$\lambda_1 = \beta_0$$

$$\lambda_2 = -\beta_0 + 2\beta_1$$

$$\lambda_3 = \beta_0 - 6\beta_1 + 6\beta_2$$

$$\lambda_4 = -\beta_0 + 12\,\beta_1 - 30\,\beta_2 + 20\,\beta_3 \qquad (2.2)$$

Hosking (1995) defined two variants of L-moments definition, which used with right-censored data as follows:

For Type – A PPWM $$\beta_r^A = \frac{1}{c^{r+1}} \int_0^c x(F)\ F^r\ dF, \qquad (2.3)$$

For Type – B PPWM $$\beta_r^B = \int_0^c x(F)\ F^r\ dF + \frac{1-c^{r+1}}{r+1}\ x(c)$$
(2.4)

In this section, different types PPWM of Weibull distribution are introduced which used to determine L-moments of a Type-I censored data, for right and left censoring respectively.

2.1 PPWM for Right Censoring for Weibull distribution

Let $x_1, x_2, \ldots, x_n$ be a Type-I censored random sample of size n with distribution function $F(x)$ and quantile function $x(u)$. Let the threshold T satisfy $F(T) = c$ and c is the fraction of observed data. Type-I right censoring occurs when the observations above the fixed threshold τ are censored. This section is concerned with two different types PPWM of Weibull distribution used in L-moments for Type-I right censored data, Type-A PPWM and Type-B PPWM respectively.

- **For Type-A PPWM**

Substituting equation (2.3) by the quantile of Weibull distribution, the r^{th} Type-A PPWM population for Weibull distribution is:

$$\beta_r^A = \frac{a}{c^{r+1}} \int_0^c [-\log\ (1-u)]^{\frac{1}{b}}\ u^r\ du$$

Put $z = -\log(1-u) \Rightarrow e^{-z} = 1-u$ this led to $-e^{-z}dz = -du$, and, $0 < u < c$

$\Rightarrow 0 < z < -\log(1-c)$. Thus,

$$\beta_r^A = \frac{a}{c^{r+1}} \int_0^{-\log(1-c)} z^{\frac{1}{b}} \,(1-e^{z})^r\, e^{-z}\, dz \qquad (2.5)$$

Substituting, $r = 0, 1$ in equation (2.5), the first two PPWM for Type-I right censoring with Type-A for Weibull distribution are:

$$\beta_0^A = \frac{a}{c} \int_0^{-\log(1-c)} z^{\frac{1}{b}}\, e^{-z}\, dz$$

$$= \frac{a}{c}\left[\int_0^{\infty} z^{\frac{1}{b}}\, e^{-z}\, dz - \int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}}\, e^{-z}\, dz\right]$$

$$= \frac{a}{c}\left[\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(-\log(1-c), \frac{1}{b}+1\right)\right]$$

and,

$$\beta_1^A = \frac{a}{c^2} \int_0^{-\log(1-c)} z^{\frac{1}{b}}\,(1-e^{-z})\, e^{-z}\, dz$$

$$= \frac{a}{c^2}\left[\int_0^{-\log(1-c)} z^{\frac{1}{b}}\, e^{-z}\, dz - \int_0^{-\log(1-c)} z^{\frac{1}{b}}\, e^{-2z}\, dz\right]$$

Putting $y = 2z \Rightarrow z = \frac{y}{2}$, this is led to $dz = \frac{dy}{2}$ and, $0 < z < -\log(1-c) \Rightarrow$

$0 < y < -2 \log(1 - c)$, thus:

$$\beta_1^A = \frac{a}{c^2}\left[\int_0^{-\log(1-c)} z^{\frac{1}{b}} e^{-z}\, dz - \int_0^{-2\log(1-c)} (\frac{y}{2})^{\frac{1}{b}} e^{-y} \frac{dy}{2}\right]$$

$$= \frac{a}{c^2}\left[\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(-\log(1-c), \frac{1}{b}+1\right) - 2^{-\frac{1+b}{b}}\left(\Gamma\left(\frac{1}{b}+1\right)\right) - \Gamma\left(-2\log(1-c), \frac{1}{b}+1\right)\right] \tag{2.6}$$

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- For Type-B PPWM

Substituting equation (2.4) by the quantile of Weibull distribution, the r^{th} Type-A PPWM population for Weibull distribution is:

$$\beta_r^B = a\int_0^c [-\log(1-u)]^{\frac{1}{b}} u^r\, du + \frac{a(1-c^{r+1})}{r+1}[-\log(1-c)]^{\frac{1}{b}}$$

$$= a\int_0^{-\log(1-c)} z^{\frac{1}{b}}(1-e^{-z})^r e^{-z}\, dz + \frac{a(1-c^{r+1})}{r+1}[-\log(1-c)]^{\frac{1}{b}} \tag{2.7}$$

Substituting, r = 0; 1 in equation (2.7), the first two PPWM for Type-I right censoring

with Type-B for Weibull distribution are:

$$\beta_0^B = a\int_0^{-\log(1-c)} z^{\frac{1}{b}} e^{-z}\, dz + a(1-c)[-\log(1-c)]^{\frac{1}{b}}$$

$$= a\left[\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(-\log(1-c), \frac{1}{b}+1\right)\right] + a(1-c)[-\log(1-c)]^{\frac{1}{b}} \tag{2.8}$$

and,

$$\beta_1^B = a\int_0^{-\log(1-c)} z^{\frac{1}{b}}(1-e^{-z})^r e^{-z}\, dz + \frac{a(1-c^2)}{2}[-\log(1-c)]^{\frac{1}{b}}$$

$$= a\left[\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(-\log(1-c), \frac{1}{b}+1\right) - 2^{-\frac{1+b}{b}}\left(\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(-2\log(1-c), \frac{1}{b}+1\right)\right)\right]$$

$$\frac{a\,(1-c^2)}{2}\left[-\log\,(1-c)\right]^{\frac{1}{b}} \qquad (2.9)$$

2.2 PPWM for Left Censoring for Weibull distribution

Type-I left censoring occurs when the observations below the fixed threshold τ are censored. Zafirakou-Koulouris *et al.* (1998) derived PPWM for left censoring, following the same approach introduced by Hosking (1995) for right censoring where:

$$For\ Type - \acute{A}\ PPWM \qquad \beta_r^{A} = \frac{1}{(1-c)^{r+1}} \int_c^1 x(F)\ (F-c)^r \quad dF, \qquad r = 0, 1, 2, \dots\dots$$
(2.10)

$$For\ Type - \acute{B}\ PPWM \qquad \beta_r^{A} = x(c)\ \frac{c^{r+1}}{(1-c)^{r+1}} + \int_c^1 F^r\ \ x(F) \quad dF, \qquad r = 0, 1, 2, \dots\dots$$
(2.11)

This section is concerned with two different types PPWM of Weibull distribution used in L-moments for Type-I left censored data, Type-A' and Type-B' PPWM for left censoring respectively.

. For Type-A' PPWM

Substituting equation (2.10) by the quantile of Weibull distribution, the r^{th} Type-A' PPWM population for Weibull distribution is:

$$\beta_r^{A} = \frac{a}{(1-c)^{r+1}} \int_c^1 [-\log\,(1-u)]^{\frac{1}{b}}\ (u-c)^r \quad du \qquad (2.12)$$

Substituting, $r = 0, 1$ in equation (2.12), the first two PPWM for Type-I left censoring with Type-A' for Weibull distribution are:

$$\beta_0^{A} = \frac{a}{(1-c)} \int_c^1 [-\log\,(1-u)]^{\frac{1}{b}} \quad du$$

Put z = $-\log\,(1-u) \Rightarrow e^{-z} = 1-u$, this is led to e^{-z}dz = $-du$

and, $c < u < 1 \Rightarrow -\log(1-c) < z < \infty$, thus:

$$\beta_0^A = \frac{a}{(1-c)} \int_{-\log(1-c)}^{\infty} [z]^{\frac{1}{b}} \, e^{-z} \, dz$$

$$= \frac{a}{1-c} \, \Gamma\left(-\log(1-c), \frac{1}{b}+1\right) \qquad (2.13)$$

Similarly, as β_1^A computed in equation (2.6), it is found that:

$$\beta_1^A = \frac{a}{(1-c)^2} \left[(1-c)\int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} \, e^{-z} \, dz - \int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} \, e^{-2z} \, dz\right]$$

Put, $y = 2z \Rightarrow z = \frac{y}{2}$, this is led to $dz = \frac{dy}{2}$ and, $-\log(1-c) < z < \infty \Rightarrow \log(1-c) < y < \infty$ thus:

$$\beta_1^A = \frac{a}{(1-c)^2} \left[(1-c)\int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} \, e^{-z} \, dz - \int_{-2\log(1-c)}^{\infty} \left(\frac{y}{2}\right)^{\frac{1}{b}} \, e^{-y} \, \frac{dy}{2}\right]$$

$$= \frac{a}{(1-c)^2} \left[(1-c)\Gamma\left(-\log(1-c), \frac{1}{b}+1\right) - 2^{-\frac{1+b}{b}}\Gamma\left(-2\log(1-c), \frac{1}{b}+1\right)\right] \qquad (2.14)$$

- **For Type-B' PPWM**

Substituting equation (2.11) by the quantile of Weibull distribution, the r^{th} Type-B' PPWM population for Weibull distribution is:

$$\beta_r^B = a\int_c^1 [-\log(1-u)]^{\frac{1}{b}}\, u^r\, du + \frac{a\,(c^{r+1})}{r+1}[-\log(1-c)]^{\frac{1}{b}}$$

$$= a\int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}}(1-e^{-z})^r e^{-z}\, dz + \frac{a\,(c^{r+1})}{r+1}[-\log(1-c)]^{\frac{1}{b}} \qquad (2.15)$$

Substituting, r = 0; 1 in equation (2.15), the first two PPWM for Type-I left

censoring with Type- B' for Weibull distribution is:

$$\beta_0^B = a\int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} e^{-z}\, dz + a\,c\,[-\log(1-c)]^{\frac{1}{b}}$$

$$= a\,\Gamma\left(-\log(1-c), \frac{1}{b}+1\right) + a\,c\,[-\log(1-c)]^{\frac{1}{b}} \qquad (2.16)$$

and,

$$\beta_1^B = a\int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}}(1-e^{-z})^r e^{-z}\, dz + \frac{a\,(c^2)}{2}[-\log(1-c)]^{\frac{1}{b}}$$

$$= a\left[\Gamma\left(-\log(1-c), \frac{1}{b}+1\right) - 2^{-\frac{1+b}{b}}\,\Gamma\left(-2\log(1-c), \frac{1}{b}+1\right)\right] + \frac{a\,(c^2)}{2}[-\log(1-c)]^{\frac{1}{b}}$$

(2.17)

3 *Direct* L-moments for censored data for Weibull distribution

Mahmoud *et al.* (2017) introduced the concept of *Direct* L-moments. They applied their method to Kumaraswamy distribution and compared it with L-moments via PPWM method and ML method. They introduced two variants method which are used with both right and left censored data respectively. In this section, *Direct* L-moment is applied to Weibull distribution as the same manner of

Kumaraswamy distribution for right and left censored data respectively.

Let $x_1, x_2, \ldots, x_n$ be a Type-I censored random sample of size n from a distribution with distribution function $F(x)$ and quantile function $x(u)$. Let the threshold T satisfy $F(T) = c$ and c is the fraction of observed data, where:

$$\underbrace{\boldsymbol{x_{1:n} \le x_{2:n} \le \cdots \le x_{m:n}}}_{\boldsymbol{m}\ (\textit{observed data})} \le \boldsymbol{T} \le \underbrace{\boldsymbol{x_{m+1:n} \le \cdots \le x_{n-1:n} \le x_{n:n}}}_{\boldsymbol{n\text{-}m}\ (\textit{censored data})}$$

- **For Type-AD**

Mahmoud *et al.* (2017) defined a Type-AD right censoring Direct L-moments, where:

$$\mu_r^A = \frac{1}{r\,c^r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{r!}{(r-k-1)!\,k!}\int_0^c x(u)\,u^{r-k-1}(c-u)^k\,du \qquad (3.1)$$

From equation (3.1), the r[th] Type-AD L-moments population for the Weibull distribution is:

$$\mu_r^A = \frac{a}{r\,c^r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{r!}{(r-k-1)!\,k!}\int_0^c[-\log\,(1-c)]^{\frac{1}{b}}\,u^{r-k-1}(c-u)^k\,du \qquad (3.2)$$

Substituting, r = 1; 2 in equation (3.2), the first two L-moments for Type-I right censoring

with Type-AD for Weibull distribution, it is found that:

$$\mu_1^A = \frac{a}{c}\int_0^c[-\log\,(1-u)]^{\frac{1}{b}}\,du$$

$$= \frac{a}{c}\int_0^{-\log(1-c)} z^{\frac{1}{b}}\,e^{-z}\,dz$$

$$= \frac{a}{c} \left[\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(-\log(1-c), \frac{1}{b}+1\right)\right] \qquad (3.3)$$

and,

$$\mu_2^A = \frac{a}{2c^2} \sum_{k=0}^{1} (-1)^k \binom{1}{k} \frac{2!}{(1-k)!\,k!} \int_0^c [-\log(1-c)]^{\frac{1}{b}} u^{1-k} (c-u)^k \, du \quad .$$

$$= \frac{a}{2c^2} \left[2\int_0^c [-\log(1-u)]^{\frac{1}{b}} \, u \, du - 2\int_0^c [-\log(1-u)]^{\frac{1}{b}} (c-u) \, du \right]$$

$$= \frac{a}{c^2} \left[(2-c)\Gamma\left(\frac{1}{b}+1\right) - (2-c)\Gamma\left(-\log(1-c), \frac{1}{b}+1\right) - 2^{-\frac{1}{b}}\left(\Gamma\left(\frac{1}{b}+1\right)\right) - \Gamma\left(-2\log(1-c), \frac{1}{b}+1\right) \right]$$

(3.4)

The standard method to compute L-moments estimator is equating the sample L-moments (M_r^A) with the corresponding population L-moments (μ_r^A). Type-AD L-moments estimators for right censored data given by:

$$M_r^A = \frac{1}{r\binom{m}{r}} \sum_{i=1}^{m} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-k-1} \binom{m-i}{k} X_{i:n} \qquad (3.5)$$

- **For Type-BD**

Mahmoud *et al*. (2017) defined a Type-BD right censoring Direct L-moments as:

$$\mu_r^B = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!\,k!} \int_0^c x(u)\, u^{r-k-1} (1-u)^k \, du + x(u)[\beta(r-k, k+1) - \beta(c, r-k, k+1)]$$

(3.6)

From equation (3.6), the r^th Type-BD L-moments population for the Weibull distribution as follows:

$$\mu_r^B = \frac{a}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{r!}{(r-k-1)!\,k!}\int_0^c[-log(1-u)]^{\frac{1}{b}}u^{r-k-1}(1-u)^k\,du + [-log(1-u)]^{\frac{1}{b}}[\beta(r-k,k+1)-\beta(c,r-k,k+1)] \tag{3.7}$$

Substituting, r = 1; 2 in equation (3.7), the first two L-moments for Type-I right censoring with Type-BD for Weibull distribution, it is found that:

$$\mu_1^B = a\left[\Gamma\left(\frac{1}{b}+1\right)-\Gamma\left(-\log(1-c),\frac{1}{b}+1\right)\right]+a(1-c)[-log(1-u)]^{\frac{1}{b}} \tag{3.8}$$

And,

$$\mu_2^B = a\left[\Gamma\left(\frac{1}{b}+1\right)-\Gamma\left(-\log(1-c),\frac{1}{b}+1\right)\right]-2^{-\frac{1}{b}}\left[\Gamma\left(\frac{1}{b}+1\right)-\Gamma\left(-2\log(1-c),\frac{1}{b}+1\right)\right]+a(1-c)[-log(1-c)]^{\frac{1}{b}} \tag{3.9}$$

The standard method to compute L-moments estimator is equating the rth sample L-moments (M_r^A) with the corresponding population L-moments (μ_r^B) Type-BD L-moments estimators are computed from the complete sample, where *n* - *m* censored data are replaced by the censoring threshold T given by:

$$M_r^B = \frac{1}{r\binom{n}{r}}\left[\sum_{i=1}^{m}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}X_{i:n} + \left[\sum_{i=m+1}^{n}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}T\right]\right] \tag{3.10}$$

3.1 **Direct *L-moments for left Censored Data***

Let $x_1, x_2, \ldots, x_n$ be a random sample of size *n*. Type-I left censoring occurs when the observations below the fixed threshold τ are censored, where:

$$\underbrace{x_{1:n} \le x_{2:n} \le \cdots \le x_{s:n}}_{s\ (censored\ data)} \le T \le \underbrace{x_{s+1:n} \le \cdots \le x_{n-1:n} \le x_{n:n}}_{n-s\ (observed\ data)}$$

Let the threshold τ satisfy $F(\tau) = h$,where, *h* is the fraction of censored data.

- **For Type-$\grave{A}D$**

Mahmoud et al. (2017) defined a Type-$\grave{A}D$ left censoring *Direct* L-moments, where:

$$\mu_r^{\grave{A}D} = \frac{1}{r(1-h)^r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!\,k!} \int_h^1 x(u)\,(u-h)^{r-k-1} (1-u)^k \, du \quad (3\text{-}11)$$

Substituting the from equation (1.3), the r^th^ Type-A'D L-moments population for the Weibull distribution is:

$$\mu_r^{\grave{A}D} = \frac{a}{r(1-h)^r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!\,k!} \int_h^1 [-\log(1-u)]^{\frac{1}{b}} (u-h)^{r-k-1} (1-u)^k \, du$$

(3-12)

Substituting, r = 1; 2 in equation (3.12), the first two L-moments for Type-I left censoring with Type-A'D for Weibull distribution are:

$$\mu_1^{\grave{A}D} = \frac{a}{1-h} \int_h^1 [-\log(1-u)]^{\frac{1}{b}} \, du$$

Put z = $-\log(1-u) \Rightarrow e^{-z} = 1-u$, this is led to e^{-z}dz = $-du$ and, h < u < 1 ⇒

- log(1 - h) < z < ∞, thus:

$$\mu_1^{\grave{A}D} = \frac{a}{1-h} \int_{-\log(1-h)}^{\infty} [z]^{\frac{1}{b}} \, e^{-z} \, dz$$

$$\mu_1^{\grave{A}D} = \frac{a}{1-h} \Gamma\left(-\log(1-h), \frac{1}{b}+1\right) \quad (3\text{-}13)$$

And,

$$\mu_2^{\grave{A}D} = \frac{a}{(1-h)^2} \left[(1-h)\Gamma\left(-\log(1-h), \frac{1}{b}+1\right) - 2^{-\frac{1}{b}}\Gamma\left(-2\log(1-h), \frac{1}{b}+1\right)\right]$$

(3-14)

The standard method to compute L-moments estimator is equating the r^th^ sample L-moments ($M_r^{\grave{A}D}$) with the corresponding population

L-moments ($\mu_r^{\grave{A}D}$) Type-A'D L-moments estimators for left censored data given by:

$$M_r^{\grave{A}D} = \frac{1}{r\binom{n-s}{r}} \sum_{i=1}^{n-s} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-k-1} \binom{n-s-i}{k} X_{s+i:n} \quad (3.15)$$

- **For Type-$\grave{B}D$**

Mahmoud et al. (2017) defined a Type-$\grave{B}D$ left censoring *Direct* L-moments, where:

$$\mu_r^{\grave{B}D} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!\,k!} \left[x(h)\,\beta(h, r-k, k+1) \int_h^1 x(u)\,(u)^{r-k-1} (1-u)^k \, du \right]$$

(3-16)

Substituting the from equation (1.3), the r[th] Type-B'D L-moments population for Weibull distribution is:

$$\mu_r^{\grave{B}D} = \frac{a}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!\,k!} \left[[-\log(1-h)]^{\frac{1}{b}} \beta(h, r-k, k+1) + \int_{-\log(1-h)}^{1} [z]^{\frac{1}{b}} (1 - e^{-z})^{r-k-1} (e^{-z})^k e^{-z} \, dz \right]$$

(3-17)

Substituting, r = 1; 2 in equation (3.17), the first two L-moments for Type-I left censoring with Type-B'D for Weibull distribution are:

$$\mu_1^{\grave{B}D} = a\,\Gamma\left(-\log(1-h), \frac{1}{b}+1\right) + a\,h\,[-\log(1-h)]^{\frac{1}{b}} \quad (3\text{-}18)$$

And,

$$\mu_2^{\grave{B}D} = a\left[h(1-h)[-\log(1-h)]^{\frac{1}{b}} + \Gamma\left(-\log(1-h), \frac{1}{b}+1\right) - 2^{-\frac{1}{b}}\Gamma\left(-2\log(1-h), \frac{1}{b}+1\right)\right]$$

(3-19)

The standard method to compute L-moments estimator is equating the r[th] sample L-moments ($M_r^{\grave{B}D}$) with the corresponding population

L-moments ($\mu_r^{\dot{B}D}$) Type-B'D L-moments estimators for left censored data given by:

$$M_r^{BD} = \frac{1}{r\binom{n}{r}}\left[\sum_{i=1}^{s}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}\,\tau + \sum_{i=s+1}^{n}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}\,X_{i:n}\right] \tag{3.20}$$

4 Maximum likelihood method for censored data for Weibull distribution

4.1 ML method for right censored data for Weibull distribution

Consider $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$ be the order statistics of size of n of Weibull distribution, Type-I right censoring occurs when m of these data are observed ($m \leq n$) and the remaining $n - m$ are censored above a known threshold T. The likelihood function of the parameters (a and b) of Weibull distribution, based on Type-I right censoring, is given by:

$$L(a, b; x) \propto \prod_{i=1}^{m} f(x_i)\,[1 - F(T)]^{n-m} \tag{4.1}$$

Therefore, the log-likelihood function is:

$$l = \log L(a, b; x) = \sum_{i=1}^{m} \log f(x_i) + (n-m)\log[1 - F(T)] \tag{4.2}$$

To estimate the unknown parameters, a and b, the first partial derivations of the log likelihood function, l, with respect to a and b respectively is needed. Setting $\frac{\partial l}{\partial a} = 0$ and $\frac{\partial l}{\partial b} = 0$, we get the likelihood equations:

$$\sum_{i=1}^{m}\left[\frac{1}{f(x_i)} * \frac{\partial f(x_i)}{\partial a}\right] + (n-m)\left[\frac{1}{1-F(T)} * \frac{\partial(1-F(T))}{\partial a}\right] = 0 \tag{4.3}$$

$$\sum_{i=1}^{m}\left[\frac{1}{f(x_i)} * \frac{\partial f(x_i)}{\partial b}\right] + (n-m)\left[\frac{1}{1-F(T)} * \frac{\partial(1-F(T))}{\partial b}\right] = 0 \tag{4.4}$$

Substituting equations (4.3) and (4.4) by the pdf, cdf and deferential equations of pfd and cdf of Weibull distribution, with respect to *a* and *b* parameters, respectively, it was found that:

$$-\frac{b\,m}{a}+\sum_{i=1}^{m}\left[\frac{b}{a^{b+1}}\right]x_i^b+(n-m)\frac{b\,T^b}{a^{b+1}}=0 \quad (4.5)$$

$$\frac{m}{b}+\sum_{i=1}^{m}\left[\left(1-\left(\frac{x_i}{a}\right)^b\right)\log\left(\frac{x_i}{a}\right)\right]-(n-m)\left(\frac{T}{a}\right)^b\log\left(\frac{T}{a}\right)=0 \quad (4.6)$$

Equations (4.5) and (4.6) constitute a system of two nonlinear equations must be solved in a and b to get the MLEs for right censored data of these parameters. It is obvious that the system of nonlinear equation has no closed form solution. So, a numerical technique is required to get the estimates of the unknown parameters.

4.2 ML method for left censored data for Weibull distribution

Consider $x_{1:n} \leq x_{2:n} \leq \ldots\ldots \leq x_{n:n}$ be the order statistics of size of *n*, Type-I left censoring occurs when the observations below the fixed threshold *T* are censored:

$$L(a,b;x) \propto [F(T)]^S * \prod_{i=s+1}^{n} f(x_i) \quad (4.7)$$

Therefore, the log-likelihood function is:

$$ll=\log L(a,b;x)= s\,log[F(T)]+\sum_{i=s+1}^{n} log f(x_i) \quad (4.8)$$

To estimate the unknown parameters, *a* and *b*, the first partial derivations of the log likelihood function, *ll*, with respect to *a* and *b* respectively is needed. Setting $\frac{\partial ll}{\partial a}=0$ and $\frac{\partial ll}{\partial b}=0$, we get the likelihood equations:

$$s\left[\frac{1}{F(T)}*\frac{\partial F(T)}{\partial a}\right]+\sum_{i=s+1}^{n}\left[\frac{1}{f(x_i)}*\frac{\partial f(x_i)}{\partial a}\right]=0 \quad (4.9)$$

$$s\left[\frac{1}{F(T)}*\frac{\partial F(T)}{\partial b}\right]+\sum_{i=s+1}^{n}\left[\frac{1}{f(x_i)}*\frac{\partial f(x_i)}{\partial b}\right]=0 \quad (4.10)$$

Substituting equations (4.9) and (4.10) by the pdf, cdf and deferential equations of pfd and cdf of Weibull distribution, with respect to *a* and *b* parameters respectively, it was found that:

$$-\frac{b}{a}\left[\frac{s\, e^{-\left(\frac{T}{a}\right)^{b}}\, T^{b}\, a^{-b}}{1-e^{-\left(\frac{T}{a}\right)^{b}}}+n-s\right]+\sum_{i=s+1}^{n}\frac{b}{a^{b+1}}\; x_i^{b} \;=\; 0 \qquad (4.11)$$

$$\left[\frac{s\, e^{-\left(\frac{T}{a}\right)^{b}}\left(\frac{T}{a}\right)^{b}\log\left(\frac{T}{a}\right)}{1-e^{-\left(\frac{T}{a}\right)^{b}}}+\frac{n-s}{b}\right]+\sum_{i=s+1}^{n}\left[1-\left(\frac{x_i}{a}\right)^{b}\right]\;\log\left(\frac{x_i}{a}\right) \;=\; 0 \qquad (4.12)$$

Equations (4.11) and (4.12) constitute a system of two nonlinear equations must be solved in a and b to get the MLEs for left censored data of these parameters. It is obvious that the system of nonlinear equation has no closed form solution. So, a numerical technique is required to get the estimates of the unknown parameters.

5 Application

In this section, a real data set is obtained to achieve the results. The data which used is the lifetime, in weeks, consists of 34 transistors in an accelerated life test, obtained from Hosking (1995) where, 3, 4, 5, 6, 6, 7, 8, 8, 9, 9, 9, 10, 10, 11, 11, 11, 13, 13, 13, 13, 13, 17, 19, 19, 25, 29, 33, 42, 42, 52, 52, 52, 52 and 53. Table (1) and (2) show ML, Direct L-moments and L-moments estimates for the two parameters of Wiebull distribution for the real data based on Type-I right censoring and Type-I left censoring, respectively.

Table 1: ML, Direct L-moments and L-moments estimates for two parameters of Wiebull distribution for the real data based on Type-I right censoring:

Estimate	ML	Direct L-moments		L-moments	
		Type-AD	Type-BD	Type-A	Type-B
$\hat{a}$	21.3769	19.9844	20.9747	19.9844	20.9747
$\hat{b}$	1.2383	1.2648	1.1109	1.2648	1.1109

Table 2: ML, Direct L-moments and L-moments estimates for the two parameters of Wiebull distribution for the real data based on

Type-I left censoring:

Estimate	ML	Direct L-moments		L-moments	
		Type-AD	Type-BD	Type-A	Type-B
$\hat{a}$	20.5384	19.7563	20.3857	19.7563	20.3857
$\hat{b}$	1.2849	1.1649	1.2391	1.1649	1.2391

6. Simulation Study

This section is devoted to present the modification of L-moments method (namely: Direct L- moments) in estimation process using a comparative numerical study. The two unknown parameters of Weibull distribution are estimated using Direct L-moments method, L-moments via PPWM method and ML method for Type-I censored data (right and left censoring). A comparative study based on relative bias (RB) and root of mean square errors. All computations are performed using Mathematica- 10 programs. The simulation study is conducted according to the following steps:

1. Generate n random sample sizes (30, 50 and 100) drawn randomly from Weibull distribution with some different values of parameters.
2. The generated data is ordered.
3. Applying formulas mentioned in (3.5), (3.10), (3.15) and (3.20).
4. Equate step (4) with the corresponding population moments to get the estimates $\hat{a}$ and $\hat{b}$.
5. The simulation process repeated 5000 times.
6. The simulation results are reported in Table(3) to Table (8), respectively.

7. Concluding Remarks

According to simulation study, the results of estimating the two unknown parameters of Weibull distribution and their characteristics are obtained from Table 3 to Table 8, respectively. From the results, it is observed that:

- As expected, the relative bias and RMSE decreases as sample sizes increases with reasonable results obtained starting from n = 50.

- The results suggest that the Direct L-moments method is better, in terms of accuracy and precision, than L-moments via PPWM and ML methods.
- In all results, relative bias and root mean square errors decrease as sample sizes increase.
- In the case of right and left censoring, the estimates of Direct L-moments method with Type-AD estimates are very close to L-moments with Type-A estimates.
- In the case of right and left censoring, the estimates of Direct L-moments method with Type-BD estimates are much more accurate than L-moments with Type-B estimates.
- In general the variance of estimates is small and this helps to obtain short confidence interval.
- It is recommended to use the Direct L-moments method with Type-AD estimates in the case of right and left censoring.

Table 3: The estimates, relative bias (RB) and RMSE for two parameters of Wiebull distribution using Direct L-moments, L-moments and ML method based on right censoring (a= 0.5 and b=5)

Type (A)											
C	*n*	Par.	Direct L-moments (Type-AD)			L-moments (Type-A)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	0.5000	0.0119	0.5041	0.5040	0.0128	0.5041	0.5032	0.0064	0.5044
		b	5.3543	0.0728	5.7071	5.3534	0.0726	5.7071	5.3550	0.0710	5.5714
	50	a	0.5010	0.0022	0.5030	0.5000	0.0019	0.5020	0.5087	0.0018	0.5062
		b	5.2825	0.0550	5.4520	5.2840	0.0586	5.4560	5.2649	0.0596	5.3850
	100	a	0.5040	0.0074	0.5049	0.5030	0.0051	0.5049	0.5090	0.0054	0.5062
		b	5.1440	0.0283	5.2180	5.1420	0.0247	5.2140	5.1570	0.0342	5.2172
20%	30	a	0.5069	0.0074	0.5025	0.5070	0.0096	0.5025	0.4963	-0.0058	0.5066
		b	5.0950	0.0191	5.1960	5.0900	0.0189	5.1970	5.1445	0.0272	5.2285
	50	a	0.4987	-0.0072	0.5054	0.4987	-0.0072	0.5054	0.4929	-0.0072	0.4929
		b	5.0842	0.0118	5.1451	5.0842	0.0118	5.1451	5.1131	0.0216	5.1622
	100	a	0.5052	0.0073	0.5091	0.5052	0.0073	0.5091	0.5048	0.0096	0.5095
		b	5.0510	0.0117	5.0895	5.0520	0.0135	5.0895	5.0663	0.0186	5.0831

Type (B)											
C	*n*	Par.	Direct L-moments (Type-BD)			L-moments (Type-B)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	0.5044	0.0174	0.5013	0.5044	0.0174	0.5013	0.5032	0.0064	0.5044
		b	5.1850	0.0390	5.4090	5.1830	0.0386	5.4088	5.3550	0.0710	5.5714
	50	a	0.5060	0.0028	0.5030	0.5080	0.0051	0.5060	0.5087	0.0018	0.5062
		b	5.1612	0.0324	5.2810	5.1609	0.0317	5.2800	5.2649	0.0596	5.3850
	100	a	0.5020	0.0023	0.5030	0.5060	0.0031	0.5050	0.5090	0.0054	0.5062
		b	5.1050	0.0289	5.1640	5.1060	0.0292	5.1670	5.1570	0.0342	5.2172
20%	30	a	0.5059	0.0066	0.5039	0.5059	0.0066	0.5039	0.4963	-0.0058	0.5066
		b	5.0233	0.0068	5.1090	5.0233	0.0068	5.1091	5.1445	0.0272	5.2285
	50	a	0.5018	0.0076	0.5031	0.5018	0.0076	0.5031	0.4929	-0.0072	0.4929

		b	5.0330	0.0058	5.0841	5.0320	0.0041	5.0841	5.1131	0.0216	5.1622
	100	a	0.5072	0.0067	0.5030	0.5072	0.0067	0.5020	0.5048	0.0096	0.5095
		b	5.0370	0.0054	5.0580	5.0350	0.0049	5.0570	5.0663	0.0186	5.0831

Table 4: The estimates, relative bias (RB) and RMSE for two parameters of Wiebull distribution using Direct L-moments, L-moments and ML method based on right censoring (a=2 and b=4)

Type (A)											
c	*n*	Par.	Direct L-moments (Type-AD)			L-moments (Type-A)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	2.0200	0.0140	2.0400	2.0260	0.0196	2.0450	2.0147	0.0073	2.0222
		b	4.2870	0.0729	4.4565	4.2890	0.0774	4.4590	4.3056	0.0764	4.4479
	50	a	2.0100	0.0095	2.0260	2.0110	0.0095	2.0200	2.0110	0.0055	2.0154
		b	4.1600	0.0455	4.2620	4.1620	0.0499	4.2640	4.1857	0.0464	4.2703
	100	a	2.0046	0.0032	2.0055	2.0055	0.0076	2.0060	2.0028	0.0014	2.0046
		b	4.0611	0.0178	4.1049	4.0604	0.0160	4.1048	4.0761	0.0190	4.1124
20%	30	a	2.0187	0.0033	2.0177	2.0187	0.0033	2.0177	2.0083	0.0041	2.0113
		b	4.1190	0.0223	4.1984	4.1180	0.0219	4.1984	4.1652	0.0413	4.2340
	50	a	2.0016	0.0058	2.0096	2.0025	0.0062	2.0096	2.0014	0.0007	2.0031
		b	4.0750	0.0176	4.1250	4.0740	0.0154	4.1255	4.0914	0.0228	4.1336
	100	a	2.0023	0.0024	2.0034	2.0023	0.0024	2.0034	2.0010	0.0005	2.0019
		b	4.0570	0.0129	4.0710	4.0560	0.0107	4.0705	4.0653	0.0163	4.0860
Type (B)											
c	*n*	Par.	Direct L-moments (Type-BD)			L-moments (Type-B)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	2.0326	0.0130	2.0414	2.0330	0.0152	2.0414	2.0147	0.0073	2.0222
		b	4.1660	0.0490	4.3070	4.1650	0.0488	4.3060	4.3056	0.0764	4.4479
	50	a	2.0238	0.0168	2.0290	2.0247	0.0173	2.0260	2.0110	0.0055	2.0154
		b	4.0980	0.0240	4.1820	4.0970	0.0218	4.1805	4.1857	0.0464	4.2703
	100	a	2.0030	0.0064	2.0000	2.0070	0.0087	2.0070	2.0028	0.0014	2.0046
		b	4.0339	0.0059	4.0600	4.0343	0.0060	4.0610	4.0761	0.0190	4.1124
20%	30	a	2.0165	0.0009	2.0162	2.0165	0.0009	2.0162	2.0083	0.0041	2.0113
		b	4.0509	0.0157	4.1211	4.0509	0.0157	4.1211	4.1652	0.0413	4.2340
	50	a	2.0041	0.0085	2.0086	2.0041	0.0085	2.0086	2.0014	0.0007	2.0031
		b	4.0270	0.0041	4.0786	4.0260	0.0039	4.0786	4.0914	0.0228	4.1336
	100	a	2.0049	0.0024	2.0003	2.0049	0.0024	2.0003	2.0010	0.0005	2.0019
		b	4.0305	0.0058	4.0570	4.0305	0.0058	4.0550	4.0653	0.0163	4.0860

Table 5: The estimates, relative bias (RB) and RMSE for two parameters of Wiebull distribution using Direct L-moments, L-moments and ML method based on right censoring (a=0.2 and b=0.8)

Type (A)											
c	*n*	Par.	Direct L-moments (Type-AD)			L-moments (Type-A)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	0.3092	0.5261	1.1028	0.3083	0.5217	1.1003	0.2238	0.1190	0.2635
		b	0.8136	0.0270	0.8609	0.8147	0.0233	0.8610	0.8653	0.0816	0.9009
	50	a	0.2289	0.1445	0.2549	0.2290	0.1451	0.2550	0.2119	0.0599	0.2249
		b	0.8063	0.0082	0.8220	0.8059	0.0027	0.8230	0.8319	0.0399	0.8488
	100	a	0.2170	0.0535	0.2149	0.2185	0.0542	0.2150	0.2051	0.0259	0.2104
		b	0.8030	0.0005	0.8170	0.8080	0.0074	0.8140	0.8217	0.0271	0.8289
20%	30	a	0.2061	0.0230	0.2136	0.2046	0.0223	0.2113	0.2035	0.0178	0.2108
		b	0.8050	0.0151	0.8370	0.8049	0.0142	0.8366	0.8381	0.0476	0.8543
	50	a	0.2050	0.0207	0.2037	0.2040	0.0211	0.2053	0.2037	0.0185	0.2086
		b	0.8010	0.0080	0.8170	0.8000	0.0066	0.8160	0.8206	0.0258	0.8294

c	n	Par.									
		a	0.2000	0.0143	0.2057	0.2080	0.0184	0.2057	0.2024	0.0120	0.2044
		b	0.7916	-0.0050	0.8040	0.7920	-0.0009	0.8030	0.8086	0.0108	0.8128
Type (B)											
c	n	Par.	Direct L-moments (Type-BD)			L-moments (Type-B)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	0.2847	0.4423	0.7137	0.2896	0.4448	0.7146	0.2238	0.1190	0.2635
		b	0.7966	-0.0067	0.8378	0.7942	-0.0098	0.8367	0.8653	0.0816	0.9009
	50	a	0.2302	0.1550	0.2581	0.2324	0.1561	0.2587	0.2119	0.0599	0.2249
		b	0.7870	-0.0112	0.8160	0.7840	-0.0154	0.8120	0.8319	0.0399	0.8488
	100	a	0.2126	0.0612	0.2162	0.2151	0.0625	0.2187	0.2051	0.0259	0.2104
		b	0.8073	0.0065	0.8145	0.8055	0.0043	0.8129	0.8217	0.0271	0.8289
20%	30	a	0.2039	0.0219	0.2183	0.2098	0.0249	0.2194	0.2035	0.0178	0.2108
		b	0.7833	-0.0133	0.8007	0.7892	-0.0125	0.8048	0.8381	0.0476	0.8543
	50	a	0.2078	0.0285	0.2000	0.2078	0.0285	0.2040	0.2037	0.0185	0.2086
		b	0.7972	-0.0011	0.8000	0.7968	-0.0053	0.8010	0.8206	0.0258	0.8294
	100	a	0.2000	0.0139	0.2000	0.2030	0.0141	0.2040	0.2024	0.0120	0.2044
		b	0.7910	-0.0044	0.7970	0.7920	-0.0030	0.7980	0.8086	0.0108	0.8128

Table 6: The estimates, relative bias (RB) and RMSE for two parameters of Wiebull distribution using Direct L-moments, L-moments and ML method based on left censoring (a=0.5 and b=5)

Type (A)											
c	n	Par.	Direct L-moments (Type-AD)			L-moments (Type-A)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	0.4953	-0.0094	0.4900	0.4960	-0.0083	0.4950	0.3156	-0.6631	0.5457
		b	5.1888	0.0376	5.2850	5.1891	0.0381	5.2880	5.6570	0.8685	5.3816
	50	a	0.4962	-0.0065	0.4937	0.4962	-0.0065	0.4937	0.4983	-0.0033	0.4986
		b	5.0988	0.0175	5.1635	5.0992	0.0184	5.1640	5.0949	0.0189	5.1610
	100	a	0.4935	-0.0054	0.4918	0.4935	-0.0054	0.4918	0.4987	-0.0024	0.4989
		b	5.0284	0.0053	5.0550	5.0284	0.0053	5.0530	5.0269	0.0053	5.0581
20%	30	a	0.4994	-0.0011	0.4913	0.4983	-0.0033	0.4913	0.4981	-0.0037	0.4984
		b	5.1351	0.0287	5.2143	5.1351	0.0287	5.2143	5.1395	0.0279	5.2182
	50	a	0.4940	-0.0091	0.4995	0.4950	-0.0089	0.4995	0.4996	-0.0006	0.4999
		b	5.0675	0.0165	5.1100	5.0675	0.0165	5.1150	5.0679	0.0135	5.1113
	100	a	0.4906	-0.0088	0.4934	0.4900	-0.0099	0.4934	0.4996	-0.0006	0.4997
		b	5.0140	0.0080	5.0362	5.0137	0.0075	5.0362	5.0156	0.0031	5.0385
Type (B)											
c	n	Par.	Direct L-moments (Type-BD)			L-moments (Type-B)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	0.4935	-0.0029	0.4954	0.4940	-0.0018	0.4954	0.3156	-0.6631	0.5457
		b	5.1228	0.0255	5.2239	5.1230	0.0260	5.2244	5.6570	0.8685	5.3816
	50	a	0.4934	-0.0033	0.4950	0.4934	-0.0033	0.4900	0.4983	-0.0033	0.4986
		b	5.0427	0.0053	5.1160	5.0419	0.0037	5.1110	5.0949	0.0189	5.1610
	100	a	0.4980	-0.0039	0.4960	0.4979	-0.0041	0.4900	0.4987	-0.0024	0.4989
		b	5.0160	0.0012	5.0471	5.0134	0.0006	5.0468	5.0269	0.0053	5.0581
20%	30	a	0.4904	-0.0018	0.4940	0.4904	-0.0018	0.4900	0.4981	-0.0037	0.4984
		b	5.0420	0.0042	5.1220	5.0440	0.0078	5.1230	5.1395	0.0279	5.2182
	50	a	0.4900	-0.0097	0.4942	0.4910	-0.0075	0.4942	0.4996	-0.0006	0.4999
		b	5.0230	0.0056	5.0700	5.0240	0.0073	5.0726	5.0679	0.0135	5.1113
	100	a	0.4900	-0.0084	0.4943	0.4930	-0.0073	0.4943	0.4996	-0.0006	0.4997
		b	4.9981	-0.0038	5.0240	4.9974	-0.0051	5.0230	5.0156	0.0031	5.0385

Table 7: The estimates, relative bias (RB) and RMSE for two parameters of Wiebull distribution using Direct L-moments, L-moments and ML method based on left censoring (a=2 and b=4)

c	n	Par.	Direct L-moments (Type-AD)			L-moments (Type-A)			ML		
Type (A)											
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	1.9847	-0.0067	1.9918	1.9858	-0.0011	1.9920	1.9874	-0.0062	1.9910
		b	4.1689	0.0423	4.2779	4.1691	0.0468	4.2780	4.1609	0.0402	4.2738
	50	a	1.9844	-0.0097	1.9965	1.9844	-0.0097	1.9965	1.9899	-0.0050	1.9919
		b	4.1140	0.0286	4.1730	4.1160	0.0290	4.1731	4.1106	0.0276	4.1722
	100	a	1.9950	-0.0074	1.9978	1.9970	-0.0063	1.9978	1.9915	-0.0042	1.9925
		b	4.0420	0.0179	4.0600	4.0460	0.0190	4.0650	4.0412	0.0103	4.0688
20%	30	a	1.9998	-0.0055	2.0025	1.9998	-0.0055	2.0025	1.9981	-0.0009	2.0005
		b	4.1594	0.0347	4.2136	4.1594	0.0347	4.2136	4.1526	0.0381	4.2179
	50	a	1.9967	-0.0064	2.0026	1.9970	-0.0053	2.0026	1.9989	-0.0005	2.0004
		b	4.0619	0.0172	4.1079	4.0620	0.0194	4.1080	4.0646	0.0161	4.1002
	100	a	1.9900	-0.0096	1.9919	1.9905	-0.0073	1.9919	1.9992	-0.0003	1.9999
		b	4.0310	0.0077	4.0488	4.0335	0.0083	4.0488	4.0314	0.0078	4.0480
Type (B)											
c	n	Par.	Direct L-moments (Type-BD)			L-moments (Type-B)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	1.9800	-0.0010	1.9810	1.9860	-0.0007	1.9840	1.9874	-0.0062	1.9910
		b	4.0420	0.0130	4.1300	4.0480	0.0144	4.1355	4.1609	0.0402	4.2738
	50	a	1.9800	-0.0053	1.9820	1.9820	-0.0042	1.9840	1.9899	-0.0050	1.9919
		b	4.0416	0.0128	4.1008	4.0429	0.0132	4.1019	4.1106	0.0276	4.1722
	100	a	1.9904	-0.0019	1.9950	1.9900	-0.0020	1.9900	1.9915	-0.0042	1.9925
		b	4.0337	0.0084	4.0596	4.0318	0.0079	4.0573	4.0412	0.0103	4.0688
20%	30	a	1.9982	-0.0013	2.0011	1.9982	-0.0013	2.0011	1.9981	-0.0009	2.0005
		b	4.1010	0.0228	4.1685	4.1030	0.0272	4.1685	4.1526	0.0381	4.2179
	50	a	1.9963	-0.0018	1.9907	1.9957	-0.0021	1.9907	1.9989	-0.0005	2.0004
		b	4.0200	0.0058	4.0506	4.0210	0.0080	4.0510	4.0646	0.0161	4.1002
	100	a	1.9961	-0.0003	1.9984	1.9961	-0.0003	1.9984	1.9992	-0.0003	1.9999
		b	4.0240	0.0097	4.0360	4.0220	0.0052	4.0340	4.0314	0.0078	4.0480

Table 8: The estimates, relative bias (RB) and RMSE for two parameters of Wiebull distribution using Direct L-moments, L-moments and ML method based on left censoring (a=0.2 and b=0.8)

c	n	Par.	Direct L-moments (Type-AD)			L-moments (Type-A)			ML		
Type (A)											
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	0.2043	0.0011	0.2007	0.2050	0.0039	0.2007	0.2040	0.0200	0.2104
		b	0.8460	0.0565	0.8569	0.8470	0.0579	0.8570	0.8506	0.0633	0.8657
	50	a	0.2048	0.0066	0.2068	0.2048	0.0066	0.2068	0.2020	0.0103	0.2057
		b	0.8250	0.0368	0.8370	0.8270	0.0371	0.8360	0.8318	0.0397	0.8409
	100	a	0.1977	-0.0051	0.2080	0.1977	-0.0051	0.2050	0.2008	0.0042	0.2027
		b	0.8102	0.0188	0.8120	0.8102	0.0188	0.8110	0.8144	0.0180	0.8185
20%	30	a	0.1927	-0.0069	0.2002	0.1938	-0.0014	0.2002	0.2034	0.0172	0.2090
		b	0.8302	0.0317	0.8462	0.8302	0.0317	0.8462	0.8467	0.0584	0.8583
	50	a	0.1958	-0.0052	0.2027	0.1958	-0.0052	0.2027	0.2005	0.0026	0.2039
		b	0.8118	0.0119	0.8270	0.8121	0.0161	0.8260	0.8200	0.0251	0.8264
	100	a	0.1927	-0.0068	0.2070	0.1930	-0.0054	0.2069	0.2006	0.0031	0.2024

		b	0.8190	0.0111	0.8192	0.8180	0.0109	0.8192	0.8126	0.0158	0.8159
Type (B)											
c	***n***	Par.	Direct L-moments (Type-BD)			L-moments (Type-B)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
50%	30	a	0.2030	0.0060	0.2053	0.2060	0.0074	0.2053	0.2040	0.0200	0.2104
		b	0.8210	0.0338	0.8439	0.8230	0.0341	0.8440	0.8506	0.0633	0.8657
	50	a	0.1980	-0.0063	0.2028	0.1930	-0.0091	0.2028	0.2020	0.0103	0.2057
		b	0.8189	0.0204	0.8277	0.8190	0.0245	0.8277	0.8318	0.0397	0.8409
	100	a	0.1970	-0.0022	0.2046	0.1940	-0.0036	0.2046	0.2008	0.0042	0.2027
		b	0.8030	0.0084	0.8130	0.8040	0.0098	0.8110	0.8144	0.0180	0.8185
20%	30	a	0.1956	-0.0013	0.2000	0.1956	-0.0013	0.2040	0.2034	0.0172	0.2090
		b	0.8219	0.0282	0.8332	0.8219	0.0282	0.8332	0.8467	0.0584	0.8583
	50	a	0.1920	-0.0144	0.2008	0.1950	-0.0130	0.2008	0.2005	0.0026	0.2039
		b	0.8030	0.0009	0.8113	0.8050	0.0036	0.8113	0.8200	0.0251	0.8264
	100	a	0.1950	-0.0028	0.2024	0.1900	-0.0055	0.2024	0.2006	0.0031	0.2024
		b	0.8080	0.0022	0.8010	0.8060	0.0019	0.8000	0.8126	0.0158	0.8159

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