Comparison of *Direct* L-moments, L-moments and ML Estimation Methods for Weibull Distribution with Type-I Censoring

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ABSTRACT

This paper presents a comparison of three different methods, Direct L-moments, L-moments via partial probability-weighted moments (PPWM) and maximum likelihood (ML) methods, respectively, to estimate the two parameters of Weibull distribution with Type-I censored data. These methods are compared in terms of estimate of the unknown parameters, relative bias and root of mean square error (RMSE) using Monte Carlo simulation to select the best method. Also, a real data set is considered to achieve the results.

Keywords: Censored Data; Estimation; Direct L-moments; L-moments; maximum likelihood; Weibull Distribution.

1- Introduction

Weibull distribution is widely used in reliability and survival analysis. It has been introduction by Wallodi Weibull 1951. This article is concerned with the two parameters Weibull distribution. The probability density function (pdf) of Weibull distribution is:

\[ f(x; a, b) = \frac{b}{a} \left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b}, \quad 0 < x < \infty, \quad a, b > 0 \] (1.1)

and the cumulative distribution function (cdf) is:

\[ F(x) = 1 - e^{-\left(\frac{x}{a}\right)^b} \] (1.2)

The quantile function can be obtained as follows:

\[ x(u) = a \left[-\log (1 - u)\right]^{1/b} \] (1.3)

This paper is concerned with estimating the two parameters \( a \) and \( b \) of Weibull distribution using three different methods, Direct L-moments with L-moments via partial probability-weighted moments (PPWM) method and maximum likelihood (ML) method, respectively with Type-I censored data.

This article is organized as follows; L-moments via PPWM for Censored Data for Weibull distribution are introduced in Section 2.
Direct L-moments for censored data for Weibull distribution is presented in Section 3. ML method for censored data for Weibull distribution is presented in Section 4. In Section 5 an application of a real data set is presented. Simulation study and concluding remarks are presented in section 6 and 7 respectively.

2- L-moments via PPWM for Censored Data for Weibull distribution


For any distribution the \( r^{th} \) L-moments is related to the \( r^{th} \) PPWM, see Hosking (1990), where:

\[
\lambda_{r+1} = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} \beta_k, \quad r = 0, 1, 2, \ldots
\]  

(2.1)

from which the first four L-moments in terms of PPWM are:

\[
\lambda_1 = \beta_0 \\
\lambda_2 = -\beta_0 + 2 \beta_1 \\
\lambda_3 = \beta_0 - 6 \beta_1 + 6 \beta_2 \\
\lambda_4 = \beta_0 - 10 \beta_1 + 20 \beta_2 - 20 \beta_3
\]
\[ \lambda_4 = -\beta_0 + 12\beta_1 - 30\beta_2 + 20\beta_3 \] (2.2)

Hosking (1995) defined two variants of L-moments definition, which used with right-censored data as follows:

\[ \text{For Type - A PPWM} \quad \beta_T^4 = \frac{1}{\alpha^3 \tau+1} \int_0^\tau x(F) F^r\, dF, \quad (2.3) \]

\[ \text{For Type - B PPWM} \quad \beta_T^2 = \int_0^\tau x(F) F^r\, dF + \frac{1-e^{\tau+1}}{r+1} x(c) \quad (2.4) \]

In this section, different types PPWM of Weibull distribution are introduced which used to determine L-moments of a Type-I censored data, for right and left censoring respectively.

2.1 **PPWM for Right Censoring for Weibull distribution**

Let \( x_1, x_2, \ldots, x_n \) be a Type-I censored random sample of size \( n \) with distribution function \( F(x) \) and quantile function \( x(u) \). Let the threshold \( T \) satisfy \( F(T) = c \) and \( c \) is the fraction of observed data. Type-I right censoring occurs when the observations above the fixed threshold \( \tau \) are censored. This section is concerned with two different types PPWM of Weibull distribution used in L-moments for Type-I right censored data, Type-A PPWM and Type-B PPWM respectively.

- **For Type-A PPWM**

Substituting equation (2.3) by the quantile of Weibull distribution, the \( r^{th} \) Type-A PPWM population for Weibull distribution is:

\[ \beta_T^4 = \frac{a}{\alpha^{r+1}} \int_0^\tau \left[-\log (1-u)\right]^{\frac{4}{\alpha}} u^r\, du \]

Put \( z = -\log(1-u) \Rightarrow e^{-z} = 1 - u \) this led to \(-e^{-z}dz = -du\), and, 0 < \( u < c \)
\[ 0 < z < -\log(1 - c). \] Thus,

\[
\beta_2^4 = \frac{a}{c^{1+1}} \int_0^{-\log(1 - c)} \frac{1}{z^b} \left(1 - e^{-z}\right)^{\frac{1}{b}} e^{-z} \, dz 
\]

Substituting, \( r = 0, 1 \) in equation (2.5), the first two PPWM for Type-I right censoring with Type-A for Weibull distribution are:

\[
\beta_2^4 = \frac{a}{c} \left[ \int_0^\infty z^\frac{1}{b} e^{-z} \, dz - \int_{-\log(1 - c)}^\infty z^\frac{1}{b} e^{-z} \, dz \right]
\]

\[ = \frac{a}{c} \left[ \Gamma\left(\frac{1}{b} + 1\right) - \Gamma\left(-\log(1 - c)\right)\Gamma\left(\frac{1}{b} + 1\right) \right] \]

and,

\[
\beta_1^4 = \frac{a}{c^2} \int_0^{-\log(1 - c)} \left(z^\frac{1}{b} \left(1 - e^{-z}\right) e^{-z} \right) \, dz
\]

\[ = \frac{a}{c^2} \left[ \left(\int_0^{-\log(1 - c)} z^\frac{1}{b} e^{-z} \, dz - \int_0^{-\log(1 - c)} z^\frac{1}{b} e^{-z} \, dz \right) \right] \]

Putting \( y = 2z \Rightarrow z = \frac{y}{2} \), this is led to \( dz = \frac{dy}{2} \) and, \( 0 < z < -\log(1 - c) \).
0 < y < \log(1 - c), thus:

\[
\beta_1^g = \frac{\alpha}{c^2} \left[ \int_0^{-\log(1-c)} \left( \frac{z}{b} \right)^{b} e^{-\frac{z}{b}} \, dz - \int_0^{\frac{\alpha}{c} \frac{\beta}{b}} \left( \frac{z}{b} \right)^{b} e^{-\frac{z}{b}} \, dy \right] \\
= \frac{\alpha}{c^2} \left[ \Gamma \left( \frac{\beta}{b} + 1 \right) - \Gamma \left( -\log(1-c), \frac{\beta}{b} + 1 \right) \right] - \Gamma(-2 \log(1-c), \frac{\beta}{b} + 1) \\
\]

(2.6)

- For Type-B PPWM

Substituting equation (2.4) by the quantile of Weibull distribution, the \( r \)th Type-A PPWM population for Weibull distribution is:

\[
\beta_1^g = \alpha \int_0^x \left[ -\log(1-u) \right]^{\frac{\beta}{b}} u^r \, du + \frac{\alpha(1-c)}{r+1} \cdot \left[ -\log(1-c) \right]^{\frac{\beta}{b}} \\
= \alpha \int_0^{-\log(1-c)} \left( \frac{1-e^{-z}}{b} \right)^{\frac{\beta}{b}} e^{-z} \, dz + \frac{\alpha(1-c)^{\frac{\beta}{b}+1}}{r+1} \cdot \left[ -\log(1-c) \right]^{\frac{\beta}{b}} \\
\]

(2.7)

Substituting, \( r = 0; 1 \) in equation (2.7), the first two PPWM for Type-I right censoring with Type-B for Weibull distribution are:

\[
\beta_1^g = \alpha \int_0^{-\log(1-c)} \left( \frac{1}{b} \right)^{\frac{\beta}{b}} e^{-z} \, dz + \alpha(1-c) \cdot \left[ -\log(1-c) \right]^{\frac{\beta}{b}} \\
= \alpha \left[ \Gamma \left( \frac{\beta}{b} + 1 \right) - \Gamma \left( -\log(1-c), \frac{\beta}{b} + 1 \right) \right] + \alpha(1-c) \cdot \left[ -\log(1-c) \right]^{\frac{\beta}{b}} \\
\]

(2.8)

\[
and,
\]

\[
\beta_1^g = \alpha \int_0^{-\log(1-c)} \left( \frac{1-e^{-z}}{b} \right)^{\frac{\beta}{b}} e^{-z} \, dz + \frac{\alpha(1-c)^2}{2} \cdot \left[ -\log(1-c) \right]^{\frac{\beta}{b}}
\]
\[
= a \left[ \Gamma \left( \frac{1}{b} + 1 \right) - \Gamma \left( -\log (1 - c), \frac{1}{b} + 1 \right) - 2^{\frac{a}{b}} \left( \Gamma \left( \frac{1}{b} + 1 \right) - \Gamma (1 - 2 \log (1 - c), \frac{1}{b} + 1) \right) \right] \\
= \frac{a (1 - c)}{z} \left[ -\log (1 - c) \right]^{\frac{1}{b}}
\]

(2.9)

2.2 PPWM for Left Censoring for Weibull distribution

Type-I left censoring occurs when the observations below the fixed threshold \( \tau \) are censored. Zafirakou-Koulouris et al. (1998) derived PPWM for left censoring, following the same approach introduced by Hosking (1995) for right censoring where:

For Type - \( \hat{A} \) PPWM \( \beta^A_r = \frac{1}{(1-c)^{r+1}} \int_c^\infty x(F) (F - c)^r dF, \quad r = 0, 1, 2, \ldots \) \( \tag{2.10} \)

For Type - \( \hat{B} \) PPWM \( \beta^B_r = x(c) \frac{c^r}{(1-c)^{r+1}} + \int_c^\infty F^r x(F) dF, \quad r = 0, 1, 2, \ldots \) \( \tag{2.11} \)

This section is concerned with two different types PPWM of Weibull distribution used in L-moments for Type-I left censored data, Type-\( \hat{A} \) and Type-\( \hat{B} \) PPWM for left censoring respectively.

For Type-\( \hat{A} \) PPWM

Substituting equation (2.10) by the quantile of Weibull distribution, the \( r \)th Type-\( \hat{A} \) PPWM population for Weibull distribution is:

\[
\beta^A_r = \frac{a}{(1-c)^{r+1}} \int_c^\infty [-\log (1 - u)]^{\frac{1}{b}} (u - c)^r du \quad \tag{2.12}
\]

Substituting, \( r = 0, 1 \) in equation (2.12), the first two PPWM for Type-I left censoring with Type-\( \hat{A} \) for Weibull distribution are:

\[
\beta^A_0 = \frac{a}{(1-c)} \int_c^\infty [-\log (1 - u)]^{\frac{1}{b}} du \quad \text{Put} \quad z = -\log (1 - u) \Rightarrow e^{-z} = 1 - u, \quad \text{this is led to} \quad e^{-z} dz = -du
\]
and, \( c < u < 1 \Rightarrow -\log(1 - c) < z < \infty \), thus:

\[
\beta_b^A = \frac{a}{(1-c)^2} \left[ (1 - c) \int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} e^{-z} \, dz - \int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} e^{-z} \, dz \right]
\]

Similarly, as \( \beta_1^A \) computed in equation (2.6), it is found that:

\[
\beta_1^A = \frac{a}{(1-c)^2} \left[ (1 - c) \int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} e^{-z} \, dz - \int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} e^{-z} \, dz \right]
\]

Put, \( y = 2z = \frac{y}{2} \), this is led to \( dz = \frac{dy}{2} \) and, \( -\log(1 - c) < z < \infty \Rightarrow \log(1 - c) < y < \infty \) thus:

\[
\beta_b^A = \frac{a}{(1-c)^2} \left[ (1 - c) \int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} e^{-z} \, dz - \int_{-\log(1-c)}^{\infty} z^{\frac{1}{b}} e^{-z} \, dz \right]
\]

\[
= \frac{a}{(1-c)^2} \left[ (1 - c)\Gamma(-\log(1-c), \frac{1}{b} + 1) - 2^{\frac{-1}{b}} \Gamma(-2\log(1-c), \frac{1}{b} + 1) \right]
\]

- **For Type-\( B' \) PPWM**

Substituting equation (2.11) by the quantile of Weibull distribution, the \({\text{i}^{th}}\) Type-B PPWM population for Weibull distribution is:
censoring with Type
Substituting, \( r = 0; 1 \) in equation (2.15), the first two PPWM for
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\[
\beta_i^2 = a \int_{c}^{1} \left[-\log (1-u) \right]^{\frac{1}{b}} u^r \, du + \frac{a (c^{r+1})}{r+1} \left[-\log (1-c) \right]^{\frac{1}{b}}
\]

\[
= a \int_{-\log (1-c)}^{\infty} z^{\frac{1}{b}} (1-e^{-z})^r e^{-z} \, dz + \frac{a (c^{r+1})}{r+1} \left[-\log (1-c) \right]^{\frac{1}{b}}
\] (2.15)

Substituting, \( r = 0; 1 \) in equation (2.15), the first two PPWM for
Type-I left
censoring with Type- \( B' \) for Weibull distribution is:

\[
\beta_i^2 = a \int_{-\log (1-c)}^{\infty} z^{\frac{1}{b}} e^{-z} \, dz + a \, c \left[-\log (1-c) \right]^{\frac{1}{b}}
\]

\[
= a \, \Gamma \left(-\log (1-c), \frac{1}{b} + 1 \right) + a \, c \left[-\log (1-c) \right]^{\frac{1}{b}}
\] (2.16)

and,

\[
\beta_i^2 = a \int_{-\log (1-c)}^{\infty} z^{\frac{1}{b}} (1-e^{-z})^r e^{-z} \, dz + \frac{a (c^{r+1})}{2} \left[-\log (1-c) \right]^{\frac{1}{b}}
\]

\[
= a \left[ \Gamma \left(-\log (1-c), \frac{1}{b} + 1 \right) - 2^{-\frac{r+1}{2}} \Gamma \left(-2 \log (1-c), \frac{1}{b} + 1 \right) \right] + \frac{a (c^{r+1})}{2} \left[-\log (1-c) \right]^{\frac{1}{b}}
\] (2.17)

3 Direct L-moments for censored data for Weibull distribution

Mahmoud et al. (2017) introduced the concept of Direct L-
moments. They applied their method to Kumaraswamy distribution and
compared it with L-moments via PPWM method and ML method. They
introduced two variants method which are used with both right and
left censored data respectively. In this section, Direct L-moment is
applied to Weibull distribution as the same manner of
Kumaraswamy distribution for right and left censored data respectively.

Let $x_1, x_2, ..., x_n$ be a Type-I censored random sample of size $n$ from a distribution with distribution function $F(x)$ and quantile function $x(u)$. Let the threshold $T$ satisfy $F(T) = c$ and $c$ is the fraction of observed data, where:

$$x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{m:n} \leq T \leq x_{m+1:n} \leq \cdots \leq x_{n-1:n} \leq x_{n:n}$$

$m$ (observed data) $n-m$ (censored data)

- For Type-AD

Mahmoud et al. (2017) defined a Type-AD right censoring Direct L-moments, where:

$$
\mu_l^k = \frac{1}{c^r} \sum_{k=0}^{r} k! \left( \frac{r-1}{k} \right) \frac{r!}{(r-k-1)!} \int_0^c x(u) u^{-r-k-1} (c-u)^k \, du \quad (3.1)
$$

From equation (3.1), the $r^{th}$ Type-AD L-moments population for the Weibull distribution is:

$$
\mu_l^k = \frac{c}{r^\alpha} \sum_{k=0}^{r} k! \left( \frac{r-1}{k} \right) \frac{r!}{(r-k-1)!} \int_0^c \left[ \log (1-c) \right]^{\frac{1}{\beta}} u^{r-k-1} (c-u)^k \, du \\
(3.2)
$$

Substituting, $r = 1; 2$ in equation (3.2), the first two L-moments for Type-I right censoring

with Type-AD for Weibull distribution, it is found that:

$$
\mu_2 = \frac{\alpha}{\beta} \int_0^c \left[ \log (1-u) \right]^{\frac{1}{\beta}} \, du \\
= \frac{\alpha}{\beta} \int_0^{\log (1-c)} z^{\frac{1}{\beta}} \, e^{-z} \, dz
$$
right censored data given by:

\[ \frac{a}{2c^2} \sum_{k=0}^{\infty} \binom{k}{r} \left( \frac{n}{k} \right) \frac{1}{(n-k-1)!} \int_0^1 \left[-\log(1-u)\right]^{\frac{1}{b}} u^{\frac{1}{b}-k} (1-u)^k \, du \]

From equation (3.6), the \( r \)th Type-BD L-moments population for the Weibull distribution as follows:

The standard method to compute L-moments estimator is equating the sample L-moments (\( M_r^A \)) with the corresponding population L-moments (\( \mu_r^A \)). Type-AD L-moments estimators for right censored data given by:

\[ M_r^A = \frac{1}{r} \sum_{k=0}^{\infty} \binom{k}{r} \left( \frac{n}{k} \right) \left( \frac{m-r}{k} \right) X_{kill} \]  

- **For Type-BD**

Mahmoud et al. (2017) defined a Type-BD right censoring Direct L-moments as:

\[ \mu_r^B = \frac{1}{r} \sum_{k=0}^{\infty} \binom{k}{r} \left( \frac{n}{k} \right) \frac{r!}{(r-k-1)!} \int_0^1 x(u) u^{k-1} (1-u)^k \, du + x(u) [\beta(r-k,k+1) - \beta(c,r-k,k+1)] \]
\[
\mu_r^F = \frac{c}{r} \sum_{k=0}^{r-1} (-1)^k \left( \frac{r-1}{k} \right) \frac{\Gamma \left( \frac{r-1}{k} + 1 \right)}{\Gamma \left( \frac{r-1}{k} + 1 \right) - \Gamma \left( \frac{r-1}{k} + 1 \right)} \left( \frac{\Gamma \left( \frac{r-1}{k} + 1 \right) - \Gamma \left( \frac{r-1}{k} + 1 \right)}{\Gamma \left( \frac{r-1}{k} + 1 \right)} - \frac{\Gamma \left( \frac{r-1}{k} + 1 \right) - \Gamma \left( \frac{r-1}{k} + 1 \right)}{\Gamma \left( \frac{r-1}{k} + 1 \right)} \right) + \left( \frac{\Gamma \left( \frac{r-1}{k} + 1 \right) - \Gamma \left( \frac{r-1}{k} + 1 \right)}{\Gamma \left( \frac{r-1}{k} + 1 \right)} - \frac{\Gamma \left( \frac{r-1}{k} + 1 \right) - \Gamma \left( \frac{r-1}{k} + 1 \right)}{\Gamma \left( \frac{r-1}{k} + 1 \right)} \right) \frac{\Gamma \left( \frac{r-1}{k} + 1 \right) - \Gamma \left( \frac{r-1}{k} + 1 \right)}{\Gamma \left( \frac{r-1}{k} + 1 \right)} \right)
\]

Substituting, \( r = 1; 2 \) in equation (3.7), the first two L-moments for Type-I right censoring with Type-BD for Weibull distribution, it is found that:

\[
\mu_1^F = a \left( \Gamma \left( \frac{1}{b} + 1 \right) - \Gamma \left( \frac{1}{b} + 1 \right) \right) + a \left( \Gamma \left( \frac{1}{b} + 1 \right) - \Gamma \left( \frac{1}{b} + 1 \right) \right) \left( \frac{\Gamma \left( \frac{1}{b} + 1 \right)}{\Gamma \left( \frac{1}{b} + 1 \right)} - \frac{\Gamma \left( \frac{1}{b} + 1 \right)}{\Gamma \left( \frac{1}{b} + 1 \right)} \right)
\]

(3.8)

And,

\[
\mu_2^F = a \left( \Gamma \left( \frac{2}{b} + 1 \right) - \Gamma \left( \frac{2}{b} + 1 \right) \right) + a \left( \Gamma \left( \frac{2}{b} + 1 \right) - \Gamma \left( \frac{2}{b} + 1 \right) \right) \left( \frac{\Gamma \left( \frac{2}{b} + 1 \right)}{\Gamma \left( \frac{2}{b} + 1 \right)} - \frac{\Gamma \left( \frac{2}{b} + 1 \right)}{\Gamma \left( \frac{2}{b} + 1 \right)} \right)
\]

(3.9)

The standard method to compute L-moments estimator is equating the \( r^{th} \) sample L-moments \((M_r^F)\) with the corresponding population L-moments \((\mu_r^F)\). Type-BD L-moments estimators are computed from the complete sample, where \( n - m \) censored data are replaced by the censoring threshold \( T \) given by:

\[
M_r^F = \frac{1}{\Gamma \left( \frac{1}{b} + 1 \right)} \left( \sum_{i=1}^{n-m} \left[ \frac{\Gamma \left( \frac{1}{b} + 1 \right) - \Gamma \left( \frac{1}{b} + 1 \right)}{\Gamma \left( \frac{1}{b} + 1 \right)} \right] \frac{\Gamma \left( \frac{1}{b} + 1 \right)}{\Gamma \left( \frac{1}{b} + 1 \right)} \right) + \frac{\Gamma \left( \frac{1}{b} + 1 \right) - \Gamma \left( \frac{1}{b} + 1 \right)}{\Gamma \left( \frac{1}{b} + 1 \right)} \right)
\]

(3.10)

### 3.1 Direct L-moments for left Censored Data

Let \( x_1, x_2, ..., x_n \) be a random sample of size \( n \). Type-I left censoring occurs when the observations below the fixed threshold \( \tau \) are censored, where:

\[
x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{s:n} \leq T \leq x_{s+1:n} \leq \cdots \leq x_{n-1:n} \leq x_{n:n}
\]

\( s \) (censored data) \hspace{1cm} \( n-s \) (observed data)

Let the threshold \( \tau \) satisfy \( F(\tau) = h \), where, \( h \) is the fraction of censored data.
For Type-D

Mahmoud et al. (2017) defined a Type-D left censoring Direct L-moments, where:

\[
\beta_{p}^{D} = \frac{1}{r(2-k)^r} \sum_{j=1}^{r} (r-1)^j \left( \frac{r-1}{k} \right) \int_{r-k+1}^{r} \chi(u) (u-h)^{r-k-1} (1-u)^k \, du (3.11)
\]

Substituting the from equation (1.3), the \( r \)th Type-A'D L-moments population for the Weibull distribution is:

\[
\mu_{p}^{D} = \frac{a}{r(2-k)^r} \sum_{j=1}^{r} (r-1)^j \left( \frac{r-1}{k} \right) \int_{r-k+1}^{r} [-\log (1-u)]^{\frac{1}{h}} (u-h)^{r-k-1} (1-u)^k \, du
\]

(3-12)

Substituting, \( r = 1; 2 \) in equation (3.12), the first two L-moments for Type-I left censoring with Type-A'D for Weibull distribution are:

\[
\mu_{1}^{D} = \frac{a}{1-h} \int_{h}^{1} [-\log (1-u)]^{\frac{1}{h}} \, du
\]

Put \( z = -\log (1-u) \Rightarrow e^{-z} = 1-u \), this is led to \( e^{-z} \, dz = -du \)

and, \( h < u < 1 \Rightarrow -\log(1-h) < z < \infty \), thus:

\[
\mu_{2}^{D} = \frac{a}{1-h} \int_{-\log(1-h)}^{\infty} \left[ z \right]^{\frac{1}{h}} e^{-z} \, dz
\]

\[
\mu_{2}^{D} = \frac{a}{1-h} \Gamma(-\log(1-h), \frac{1}{h} + 1) \quad (3-13)
\]

And,

\[
\mu_{3}^{D} = \frac{a}{(1-k)^3} \left[ (1-h)\Gamma(-\log(1-h), \frac{1}{h} + 1) - 2^{\frac{1}{h}}\Gamma(-2\log(1-h), \frac{1}{h} + 1) \right]
\]

(3-14)

The standard method to compute L-moments estimator is equating the \( r \)th sample L-moments \( \mu_{p}^{D} \) with the corresponding population
L-moments ($\mu_{r}^{BD}$) Type-A'D L-moments estimators for left censored data given by:

$$M_{r}^{BD} = \frac{1}{r(r-1)} \sum_{k=1}^{r} \sum_{l=1}^{r-k} (-1)^{k} \left(\frac{r-1}{k}\right) \left(\frac{r-1}{k} - 1\right)^{n-l} X_{r-k+l}$$

(3.15)

- For Type-$\tilde{B}D$

Mahmoud et al. (2017) defined a Type-$\tilde{B}D$ left censoring Direct L-moments, where:

$$\mu_{r}^{BD} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^{k} \left(\frac{r-1}{k}\right) \frac{r!}{(r-k-1)!} \int_{0}^{1} \frac{x^k (1-x)^{r-k-1}}{e^{(r-k-1)\theta}} e^{-\theta} dx$$

(3-16)

Substituting the from equation (1.3), the $r^{th}$ Type-B'D L-moments population for Weibull distribution is:

$$\mu_{r}^{BD} = \frac{a}{r} \sum_{k=0}^{r-1} (-1)^{k} \left(\frac{r-1}{k}\right) \frac{r!}{(r-k-1)!} \left[\log(1-h) + \gamma \log(1-h)\right]$$

(3-17)

Substituting, $r = 1; 2$ in equation (3.17), the first two L-moments for Type-I left censoring with Type-B'D for Weibull distribution are:

$$\mu_{1}^{BD} = a \Gamma\left(-\log(1-h), \frac{1}{b} + 1\right) + a h \left[-\log (1-h)\right]$$

(3-18)

And,

$$\mu_{2}^{BD} = a \left[h(1-h)\left[-\log (1-h)\right] + \Gamma\left(-\log(1-h), \frac{1}{b} + 1\right) - 2^{-\frac{1}{b}} \Gamma(-2\log(1-h), \frac{1}{b} + 1)\right]$$

(3-19)

The standard method to compute L-moments estimator is equating the $r^{th}$ sample L-moments ($M_{r}^{BD}$) with the corresponding population
L-moments ($\mu_{-1}^{LBD}$) Type-B'D L-moments estimators for left censored data given by:

$$M_{n}^{BD} \equiv \frac{1}{r} \left[ \sum_{i=1}^{r} \sum_{k=1}^{m} (-1)^{k} \binom{r-1}{k} \binom{n-r}{k-1} \sum_{i} X_{i} \right]$$

(3.20)

4 Maximum likelihood method for censored data for Weibull distribution

4.1 ML method for right censored data for Weibull distribution

Consider $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$ be the order statistics of size of $n$ of Weibull distribution, Type-I right censoring occurs when $m$ of these data are observed ($m \leq n$) and the remaining $n - m$ are censored above a known threshold $T$. The likelihood function of the parameters ($a$ and $b$) of Weibull distribution, based on Type-I right censoring, is given by:

$$L(a, b; x) \propto \prod_{i=1}^{m} f(x_i) [1 - F(T)]^{n-m}$$

(4.1)

Therefore, the log-likelihood function is:

$$l = \log L(a, b; x) = \sum_{i=1}^{n} \log f(x_i) + (n - m) \log [1 - F(T)]$$

(4.2)

To estimate the unknown parameters, $a$ and $b$, the first partial derivations of the log likelihood function, $l$, with respect to $a$ and $b$ respectively is needed. Setting $\frac{\partial l}{\partial a} = 0$ and $\frac{\partial l}{\partial b} = 0$, we get the likelihood equations:

$$\sum_{i=1}^{m} \left[ \frac{1}{f(x_i)} \frac{\partial f(x_i)}{\partial a} \right] + (n - m) \left[ \frac{1}{1 - F(T)} \frac{\partial [1 - F(T)]}{\partial a} \right] = 0$$

(4.3)

$$\sum_{i=1}^{m} \left[ \frac{1}{f(x_i)} \frac{\partial f(x_i)}{\partial b} \right] + (n - m) \left[ \frac{1}{1 - F(T)} \frac{\partial [1 - F(T)]}{\partial b} \right] = 0$$

(4.4)
Substituting equations (4.3) and (4.4) by the pdf, cdf and deferential equations of pdf and cdf of Weibull distribution, with respect to $a$ and $b$ parameters, respectively, it was found that:

$$-\frac{b n}{a} + \sum_{i=1}^{n} \left[ \frac{b}{a^{b+1}} \right] x_i^b + (n - m) \frac{b^2 x_i^b}{a^{b+1}} = 0 \tag{4.5}$$

$$\frac{b}{a} + \sum_{i=1}^{n} \left[ \left( 1 - \left( \frac{x_i}{a} \right)^b \right) \log \left( \frac{x_i}{a} \right) \right] - (n - m) \left( \frac{x_i}{a} \right)^b \log \left( \frac{x_i}{a} \right) = 0 \tag{4.6}$$

Equations (4.5) and (4.6) constitute a system of two nonlinear equations must be solved in $a$ and $b$ to get the MLEs for right censored data of these parameters. It is obvious that the system of nonlinear equation has no closed form solution. So, a numerical technique is required to get the estimates of the unknown parameters.

**4.2 ML method for left censored data for Weibull distribution**

Consider $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$ be the order statistics of size of $n$, Type-I left censoring occurs when the observations below the fixed threshold $T$ are censored:

$$L(a, b; x) \propto \left[ F(T) \right]^s \prod_{i=s+1}^{n} f(x_i) \tag{4.7}$$

Therefore, the log-likelihood function is:

$$ll = \log L(a, b; x) = s \log [F(T)] + \sum_{i=s+1}^{n} \log f(x_i) \tag{4.8}$$

To estimate the unknown parameters, $a$ and $b$, the first partial derivations of the log likelihood function, $ll$, with respect to $a$ and $b$ respectively is needed. Setting $\frac{\partial ll}{\partial a} = 0$ and $\frac{\partial ll}{\partial b} = 0$, we get the likelihood equations:

$$s \left[ \frac{1}{F(T)} \right] \frac{\partial [F(T)]}{\partial a} + \sum_{i=s+1}^{n} \left[ \frac{1}{f(x_i)} \right] \frac{\partial f(x_i)}{\partial a} = 0 \tag{4.9}$$

$$s \left[ \frac{1}{F(T)} \right] \frac{\partial [F(T)]}{\partial b} + \sum_{i=s+1}^{n} \left[ \frac{1}{f(x_i)} \right] \frac{\partial f(x_i)}{\partial b} = 0 \tag{4.10}$$
Substituting equations (4.9) and (4.10) by the pdf, cdf and deferential equations of pfd and cdf of Weibull distribution, with respect to \(a\) and \(b\) parameters respectively, it was found that:

\[
-\frac{b}{a} \left[ \frac{e^{-b} \left( \frac{x}{a} \right)^b}{1 - e^{-b}} + b - g \right] + \sum_{t=1}^{n} \frac{b}{a} \left( \frac{x_t}{a} \right)^b x_t \hat{a} = 0
\]

(4.11)

\[
\left[ \frac{b}{a} \frac{e^{-b} \left( \frac{x}{a} \right)^b \log \left( \frac{x}{a} \right)}{1 - e^{-b}} + \frac{e^{-b} \left( \frac{x}{a} \right)^b \log \left( \frac{x}{a} \right)}{1 - e^{-b}} \right] + \sum_{t=1}^{n} \left[ 1 - \left( \frac{x_t}{a} \right)^b \right] \log \left( \frac{x_t}{a} \right) = 0
\]

(4.12)

Equations (4.11) and (4.12) constitute a system of two nonlinear equations must be solved in \(a\) and \(b\) to get the MLEs for left censored data of these parameters. It is obvious that the system of nonlinear equation has no closed form solution. So, a numerical technique is required to get the estimates of the unknown parameters.

5 Application

In this section, a real data set is obtained to achieve the results. The data which used is the lifetime, in weeks, consists of 34 transistors in an accelerated life test, obtained from Hosking (1995) where, 3, 4, 5, 6, 6, 7, 8, 8, 9, 9, 10, 10, 11, 11, 11, 11, 13, 13, 13, 13, 17, 19, 19, 25, 29, 33, 42, 42, 52, 52, 52, 52 and 53. Table (1) and (2) show ML, Direct L-moments and L-moments estimates for the two parameters of Wiebull distribution for the real data based on Type-I right censoring and Type-I left censoring, respectively.

Table 1: ML, Direct L-moments and L-moments estimates for two parameters of Wiebull distribution for the real data based on Type-I right censoring:

<table>
<thead>
<tr>
<th>Estimate</th>
<th>ML</th>
<th>Direct L-moments</th>
<th>L-moments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Type-AD</td>
<td>Type-BD</td>
</tr>
<tr>
<td>(\hat{a})</td>
<td>21.3769</td>
<td>19.9844</td>
<td>20.9747</td>
</tr>
<tr>
<td>(\hat{b})</td>
<td>1.2383</td>
<td>1.2648</td>
<td>1.1109</td>
</tr>
</tbody>
</table>

Table 2: ML, Direct L-moments and L-moments estimates for the two parameters of Wiebull distribution for the real data based on
Type-I left censoring:

<table>
<thead>
<tr>
<th>Estimate</th>
<th>ML</th>
<th>Direct L-moments</th>
<th>L-moments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Type-AD</td>
<td>Type-BD</td>
</tr>
<tr>
<td>(\hat{a})</td>
<td>20.5384</td>
<td>19.7563</td>
<td>20.3857</td>
</tr>
<tr>
<td>(\hat{b})</td>
<td>1.2849</td>
<td>1.1649</td>
<td>1.2391</td>
</tr>
</tbody>
</table>

### 6. Simulation Study

This section is devoted to present the modification of L-moments method (namely: Direct L-moments) in estimation process using a comparative numerical study. The two unknown parameters of Weibull distribution are estimated using Direct L-moments method, L-moments via PPWM method and ML method for Type-I censored data (right and left censoring). A comparative study based on relative bias (RB) and root of mean square errors. All computations are performed using Mathematica-10 programs. The simulation study is conducted according to the following steps:

1. Generate \(n\) random sample sizes (30, 50 and 100) drawn randomly from Weibull distribution with some different values of parameters.
2. The generated data is ordered.
3. Applying formulas mentioned in (3.5), (3.10), (3.15) and (3.20).
4. Equate step (4) with the corresponding population moments to get the estimates \(\hat{a}\) and \(\hat{b}\).
5. The simulation process repeated 5000 times.
6. The simulation results are reported in Table(3) to Table (8), respectively.

### 7. Concluding Remarks

According to simulation study, the results of estimating the two unknown parameters of Weibull distribution and their characteristics are obtained from Table 3 to Table 8, respectively. From the results, it is observed that:

- As expected, the relative bias and RMSE decreases as sample sizes increases with reasonable results obtained starting from \(n = 50\).
The results suggest that the Direct L-moments method is better, in terms of accuracy and precision, than L-moments via PPWM and ML methods.

In all results, relative bias and root mean square errors decrease as sample sizes increase.

In the case of right and left censoring, the estimates of Direct L-moments method with Type-AD estimates are very close to L-moments with Type-A estimates.

In the case of right and left censoring, the estimates of Direct L-moments method with Type-BD estimates are much more accurate than L-moments with Type-B estimates.

In general the variance of estimates is small and this helps to obtain short confidence interval.

It is recommended to use the Direct L-moments method with Type-AD estimates in the case of right and left censoring.

**Table 3:** The estimates, relative bias (RB) and RMSE for two parameters of Weibull distribution using Direct L-moments, L-moments and ML method based on right censoring (a = 0.5 and b=5)

<table>
<thead>
<tr>
<th>Type (A)</th>
<th>C</th>
<th>n</th>
<th>Par.</th>
<th>Direct L-moments (Type-AD)</th>
<th>L-moments (Type-A)</th>
<th>ML</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Estimate</td>
<td>RB</td>
<td>RMSE</td>
</tr>
<tr>
<td>50%</td>
<td>30</td>
<td></td>
<td>a</td>
<td>0.5000</td>
<td>0.0119</td>
<td>0.5041</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>b</td>
<td>5.3543</td>
<td>0.0728</td>
<td>5.7071</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td></td>
<td>a</td>
<td>5.0100</td>
<td>0.0022</td>
<td>0.5030</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>b</td>
<td>5.2825</td>
<td>0.0550</td>
<td>5.4520</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td></td>
<td>a</td>
<td>5.0400</td>
<td>0.0074</td>
<td>0.5049</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b</td>
<td>5.1440</td>
<td>0.0283</td>
<td>5.2180</td>
<td>5.1420</td>
</tr>
<tr>
<td>20%</td>
<td>30</td>
<td></td>
<td>a</td>
<td>0.5069</td>
<td>0.0074</td>
<td>0.5025</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>b</td>
<td>5.0950</td>
<td>0.0191</td>
<td>5.1960</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td></td>
<td>a</td>
<td>0.4987</td>
<td>-0.0072</td>
<td>0.5054</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>b</td>
<td>5.0842</td>
<td>0.0118</td>
<td>5.1451</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td></td>
<td>a</td>
<td>0.5052</td>
<td>0.0073</td>
<td>0.5091</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b</td>
<td>5.0510</td>
<td>0.0117</td>
<td>5.0895</td>
<td>5.0520</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Type (B)</th>
<th>C</th>
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<th>Par.</th>
<th>Direct L-moments (Type-BD)</th>
<th>L-moments (Type-B)</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Estimate</td>
<td>RB</td>
<td>RMSE</td>
</tr>
<tr>
<td>50%</td>
<td>30</td>
<td></td>
<td>a</td>
<td>0.5044</td>
<td>0.0174</td>
<td>0.5013</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>5.1850</td>
<td>0.0390</td>
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<tr>
<td></td>
<td>50</td>
<td></td>
<td>a</td>
<td>5.0600</td>
<td>0.0028</td>
<td>0.5030</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>b</td>
<td>5.1612</td>
<td>0.0324</td>
<td>5.2810</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td></td>
<td>a</td>
<td>5.1050</td>
<td>0.0289</td>
<td>5.1640</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b</td>
<td>5.059</td>
<td>0.0066</td>
<td>5.0393</td>
<td>5.059</td>
</tr>
<tr>
<td>20%</td>
<td>30</td>
<td></td>
<td>a</td>
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<td>0.0076</td>
<td>0.5031</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>b</td>
<td>5.0233</td>
<td>0.0068</td>
<td>5.1090</td>
</tr>
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### Table 4: The estimates, relative bias (RB) and RMSE for two parameters of Weibull distribution using Direct L-moments, L-moments and ML method based on right censoring (a=2 and b=4)

<table>
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<tr>
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<th>L-moments (Type-A)</th>
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<td>RB</td>
<td>RMSE</td>
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<tr>
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<td>50%</td>
<td>a</td>
<td>2.0200</td>
<td>0.0140</td>
<td>2.0400</td>
</tr>
<tr>
<td></td>
<td>50%</td>
<td>b</td>
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<td></td>
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<td>0.0058</td>
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### Table 5: The estimates, relative bias (RB) and RMSE for two parameters of Weibull distribution using Direct L-moments, L-moments and ML method based on right censoring (a=0.2 and b=0.8)

<table>
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<th>Par.</th>
<th>Direct L-moments (Type-AD)</th>
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<th>ML</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Estimate</td>
<td>RB</td>
<td>RMSE</td>
</tr>
<tr>
<td>30</td>
<td>50%</td>
<td>a</td>
<td>0.3092</td>
<td>0.5261</td>
<td>1.0028</td>
</tr>
<tr>
<td></td>
<td>50%</td>
<td>a</td>
<td>0.8136</td>
<td>0.0270</td>
<td>0.8609</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>b</td>
<td>0.2289</td>
<td>0.1445</td>
<td>0.2549</td>
</tr>
<tr>
<td>30</td>
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<td>0.2061</td>
<td>0.0230</td>
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<td>0.0207</td>
<td>0.2037</td>
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</table>

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<table>
<thead>
<tr>
<th>c</th>
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<th>L-moments (Type-B)</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Estimate</td>
<td>RMSE</td>
<td>Estimate</td>
</tr>
<tr>
<td>20%</td>
<td>50%</td>
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<td>0.2000</td>
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<td>0.7920</td>
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<td>0.2057</td>
<td>0.0800</td>
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<td>b</td>
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<td>0.0108</td>
<td>0.8055</td>
</tr>
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</table>

**Table 6:** The estimates, relative bias (RB) and RMSE for two parameters of Weibull distribution using Direct L-moments, L-moments and ML method based on left censoring (a=0.5 and b=5)
Table 7: The estimates, relative bias (RB) and RMSE for two parameters of Weibull distribution using Direct L-moments, L-moments and ML method based on left censoring \((a=2 \text{ and } b=4)\)

<table>
<thead>
<tr>
<th>c</th>
<th>n</th>
<th>Par.</th>
<th>Direct L-moments (Type-AD)</th>
<th>L-moments (Type-A)</th>
<th>ML</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Estimate</td>
<td>RB</td>
<td>RMSE</td>
</tr>
<tr>
<td>30</td>
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<td>a</td>
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</tr>
<tr>
<td></td>
<td></td>
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<td>0.0286</td>
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<td></td>
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<td>4.0420</td>
<td>0.0179</td>
<td>4.0600</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>c</th>
<th>n</th>
<th>Par.</th>
<th>Direct L-moments (Type-BD)</th>
<th>L-moments (Type-B)</th>
<th>ML</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
<td></td>
<td>Estimate</td>
<td>RB</td>
<td>RMSE</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>a</td>
<td>1.9800</td>
<td>-0.0010</td>
<td>1.9810</td>
</tr>
<tr>
<td></td>
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<tr>
<td>50</td>
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Table 8: The estimates, relative bias (RB) and RMSE for two parameters of Weibull distribution using Direct L-moments, L-moments and ML method based on left censoring \((a=0.2 \text{ and } b=0.8)\)

<table>
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<th>L-moments (Type-A)</th>
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References


Balakrishnan (Ed.), Recent Advances in Life-Testing and Reliability (pp. 546–560). CRC


