Inverse Exponentiated Lomax Distribution: Properties and its Application

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Abstract:

A three-parameter continuous distribution, namely, inverse exponentiated Lomax distribution is proposed. The density function for the new distribution can be right-skewed, reversed-J shaped, unimodel and decreasing hazard rate function. Several properties of this distribution are discussed such as explicit expressions for quantile function, mode, mean residual life time, mean failure of time, and ordinary moments, mean residual life time, mean failure of time and the density function for the some order statistics. The maximum likelihood estimators of the parameters are derived. Simulation study is given to illustrate theoretical results. The flexibility of the new model is illustrated by real data.

Keywords: *Inverse Exponentiated Lomax distribution; censored sample data; maximum likelihood method; simulation study.*

1. Introduction

Many lifetime distributions have been created with an interpretation for applications in several areas, in specific, survival analysis, reliability engineering, demography, actuarial study, hydrology and others. Generalizing continuous univariate distributions is an old practice by introducing additional parameters such as location, scale shape and inequality in the distribution and then seeing changes in its shapes. The methods to find fitting new models for data sets which are very popular currently among the scientists becomes more flexible and more fitting for real data sets. The inverted distributions have a wide range of applications; in problems related to econometrics, biological sciences, survey sampling, engineering sciences, medical research and life testing problems. Also, it is employed in financial literature, environmental studies, survival theory, see Abd EL-Kader (2013).

Many researchers motivated the inverted distributions and its applications. Such as Folks (1983), Lehmann and Shaffer (1988), Calabria and Pulcini (1990), Khan, et al. (2008) and Khan (2010). A number of studies have showed some details about the exponentiated Lomax distribution for example Abdul-Moniem (2012), Ashour and Eltehiwy (2013), Salem (2014), El-Bassiouny et al. (2015) introduced exponentiated Lomax distribution and Ashour and Eltehiwy (2013).

The objective of this paper is to propose and study a new model distribution by inversing the exponentiated Lomax. The *inverse exponentiated Lomax distribution* (IELomax) will be more flexible and more applicable in real data.

The rest of this paper is organized as follows. Section 2 introduces IELomax model formulation. The structural characteristics of IELomax distribution including *cumulative distribution function* (cdf), *probability density function* (pdf), *hazard function* (hf) and *reliability function* (rf). Some properties of IElomax distribution are presented in Section 3. Section 4 gives the parameters estimations using maximum likelihood method. Simulation study is provided in Section 5. Section 6 discussed concludes and remarks. Finally, to more indication of new model flexibility is applied on real data in Section 7.

2.Model Formulation

A new distribution is proposed by considering t=1/y, where the random variable y follows exponentiated Lomax distribution with parameters α, θ and λ. The distribution of t is referred to as inverse exponentiated Lomax distribution. Symbolically, it is abbreviated by $t \sim IELomax\,(\alpha, \theta, \lambda)$ to indicate that the random variable t has inverse exponentiated Lomax distribution with parameters α, θ and λ.

The CDFand PDF

The CDF and PDF are given respectively by:

$$\mathbf{F}(t; \alpha, \theta, \lambda) = \left[1 - \left(1 + \frac{\lambda}{t}\right)^{-\theta}\right]^{\alpha}, \quad t > 0, \alpha, \theta, \lambda > 0, \quad (1)$$

and

$$\mathbf{f}(t; \alpha, \theta, \lambda) = \alpha\, \theta\, \lambda\, t^{-2} \left(1 + \frac{\lambda}{t}\right)^{-\theta-1} \left(1 - \left(1 + \frac{\lambda}{t}\right)^{-\theta}\right)^{\alpha-1},$$

$$t > 0, \alpha, \theta, \lambda > 0, \quad (2)$$

where α and θ are the shape parameters and λ is a scale parameter.

Figure 1, 2 and 3 show that some of the possible shapes of the pdf of the IELomax (α, θ, λ).

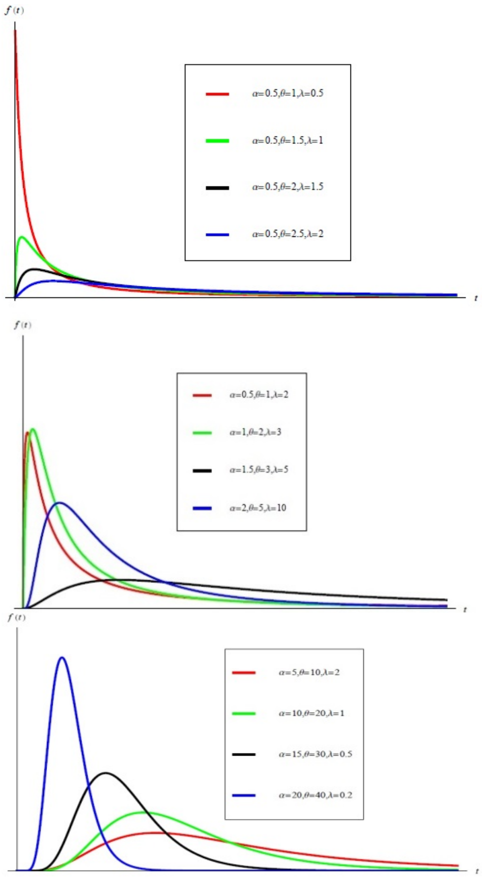

Figure 1, 2, 3 PDF of the IELomax $(\alpha, \theta, \lambda)$

Reliability function

If $t \sim IELomax\ (\alpha, \theta, \lambda)$, the RF is given as follows:

$$R(t; \alpha, \theta, \lambda) = 1 - \left[1 - (1 + \frac{\lambda}{t})^{-\theta}\right]^{\alpha}, \qquad t > 0, \alpha, \theta, \lambda > 0, \quad (3)$$

Hazard rate and reversed hazard functions

The HRF and *reversed hazard function* are given as follows, respectively

$$h(t;\alpha,\theta,\lambda)=\frac{\alpha\,\theta\,\lambda\, t^{-2}\left(1+\frac{\lambda}{t}\right)^{-\theta-1}\left(1-\left(1+\frac{\lambda}{t}\right)^{-\theta}\right)^{\alpha-1}}{1-\left[1-\left(1+\frac{\lambda}{t}\right)^{-\theta}\right]^{\alpha}},\quad t>0,\alpha,\theta,\lambda>0. \quad (4)$$

Figure 4 displays some of possible shapes of the hazard rate function of the IELomax (α,θ,λ). It is noted that the hazard function decreases, so the IELomax distribution will be more applicable in infant mortality and medicine fields.

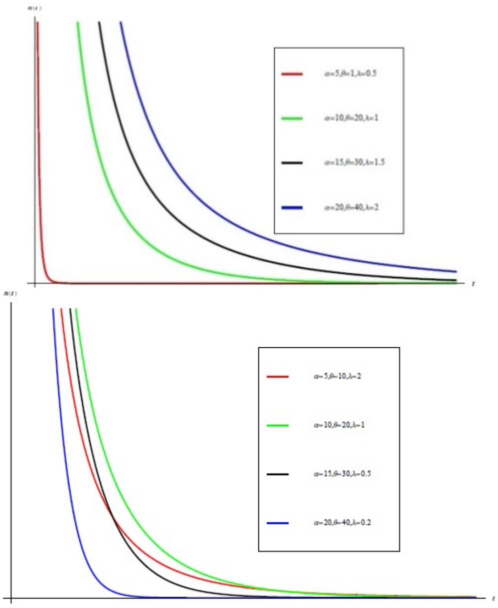

Figure 4: HRF of IELomax (α,θ,λ) for different values of parameters.

The reversed hazard function, also named retro hazard, was first mentioned by the name 'dual of the hazard rate' in Barlow *et al*. (1963). The name 'reversed hazard rate' was first used by Lagakos *et al*. (1988). According to Lagakos *et al*. (1988) the reversed hazard function extends the concept of the hazard rate to a reverse time direction and is given by

$$rh(t;\alpha,\theta,\ \lambda)=\frac{\alpha\,\theta\,\lambda\,t^{2}\left(1+\frac{\lambda}{t}\right)^{-\theta-1}\left(1-\left(1+\frac{\lambda}{t}\right)^{-\theta}\right)^{\alpha-1}}{\left[1-(1+\frac{\lambda}{t})^{-\theta}\right]^{\alpha}},\ \ t>0,\alpha,\theta,\lambda>0.\ (5)$$

3 Statistical Properties

This section is devoted to study some statistical properties for the IELomax distribution

The odd function:-

The odd function (Of) is given by

$$Of(t;\alpha,\theta,\ \lambda)=\frac{F(t)}{R(t)}=\frac{\left[1-(1+\frac{\lambda}{t})^{-\theta}\right]^{\alpha}}{1-\left[1-(1+\frac{\lambda}{t})^{-\vartheta}\right]^{\alpha}},\quad t>0,\alpha,\vartheta,\lambda>0.\quad (6)$$

The quantile function:-

The quantile function of the IELomax distribution is given by

$$t_p=\lambda\left[\left(1-p^{\frac{1}{\alpha}}\right)^{-\frac{1}{\theta}}-1\right]^{-1},\qquad t>0,\alpha,\theta,\lambda>0.\qquad (7)$$

since the median is 50% quantile, then by setting *p=0.5* in equation (7).

The inter-quantile range:-

The inter-quantile range (IQR) can be expressed as

IQR(t)=

$$\lambda\left[\left(1-0.75^{\frac{1}{\alpha}}\right)^{-\frac{1}{\theta}}-1\right]^{-1}-\lambda\left[\left(1-0.25^{\frac{1}{\alpha}}\right)^{-\frac{1}{\theta}}-1\right]^{-1},$$

$$t>0,\alpha,\theta,\lambda>0\qquad (8)$$

Bowley skewness-Moors kurtosis based on octiles:-

$$SK=\frac{Q\left(\frac{3}{4}\right)+Q\left(\frac{1}{4}\right)-2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)},\qquad (9)$$

$$KR = \frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}, \qquad (10)$$

3.5 Mode

The mode of the IELomax distribution *is given by*

$$Mode = \acute{f}(t) = 0$$

$$Mode = \frac{\alpha\theta\lambda\left(1+\frac{\lambda}{t}\right)^{-(\theta+1)}\left(1-\left(1+\frac{\lambda}{t}\right)^{-\theta}\right)^{\alpha-1}}{t^2}, \qquad (11)$$

The mean time of failure:-

The *mean time of failure* (MTTF) is given by

$$MTTF = \int_0^{\infty} R(t)dt = \sum_{j=0}^{\alpha}(-1)^{\alpha}\binom{\alpha}{j}\frac{1}{-\theta j+1}, \qquad (12)$$

The mean residual life:-

The *mean residual life* (MRL) is given as follows

$$MRL = \frac{1}{R(t)}\int_t^{\infty} R(u)du = \frac{1}{1-\left(1-\left(1+\frac{\lambda}{t}\right)^{-\theta}\right)^{\alpha}}\left[-\sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j\left(1+\frac{\lambda}{t}\right)^{-\theta\alpha}\right], (13)$$

Order Statistics:-

The i^{th} order statistic of a random sample of size n from the IELomax (α,θ,λ) distribution has the following density

$$g(t_{(i)}) = n\binom{n-1}{i-1}\alpha\,\theta\,\lambda\,t_{(i)}^{-2}\left(1+\frac{\lambda}{t_{(i)}}\right)^{-\theta-1}$$

$$\left(1-\left(1+\frac{\lambda}{t_{(i)}}\right)^{-\theta}\right)^{\alpha-i-2}\left[1-\left(1-\left(1+\frac{\lambda}{t_{(i)}}\right)^{-\theta}\right)^{\alpha}\right]^{n-i}, t_{(i)} \geq 0 \qquad (14)$$

Special cases

- When i=1, one obtain the pdf of the first order

$$G(t_{(1)}) = n\alpha\,\theta\,\lambda\,t_{(1)}{}^{-2}\left(1+\frac{\lambda}{t_{(1)}}\right)^{-\theta-1}\left(1-\left(1+\frac{\lambda}{t_{(1)}}\right)^{-\theta}\right)^{\alpha-1}$$

$$\left[1-\left(1-\left(1+\frac{\lambda}{t_{(1)}}\right)^{-\theta}\right)^{\alpha}\right]^{n-1},\quad t_{(1)} \geq 0 \qquad (15)$$

- When i=n, one obtain the pdf of the largest order

$$g(t_{(n)}) = n\alpha\,\theta\,\lambda\,t_{(n)}{}^{-2}\left(1+\frac{\lambda}{t_{(n)}}\right)^{-\theta-1}\left(1-\left(1+\frac{\lambda}{t_{(n)}}\right)^{-\theta}\right)^{\alpha-n-2},$$

$$t_{(n)} \geq 0 \qquad (16)$$

The Moments:-

Like Cauchy distribution, there are no moment , kurtosis, skewness, moment generating function, probability generating function, probability weight moment of the IELomax (α, θ, λ) distribution.

4. Estimation of The parameters, Reliability and Hazard Function

In this section, maximum likelihood method is considered to estimate the involved parameters, the asymptotic distribution of $\hat{\alpha}, \hat{\theta}$ *and* $\hat{\lambda}$ are obtained using the elements of the inverse Fisher information matrix

4.1Maximum likelihood estimation

Suppose that $t_1 < t_2 < \cdots < t_r$ be a random sample of size r from IEL$(\alpha, \theta, \lambda)$ distribution.

The likelihood function is expressed as follow

$$L(\alpha,\theta,\lambda;\underline{t}) \propto \prod_{i=1}^{r} \alpha\,\theta\,\lambda\, t_i^{\,2}\left(1+\frac{\lambda}{t_i}\right)^{-\theta-1}\left(1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right)^{\alpha-1} \left[1-\left[1-(1+\frac{\lambda}{t_r})^{-\theta}\right]^{\alpha}\right]^{n-r}, \qquad (17)$$

The natural logarithm of $L(\alpha,\theta,\lambda;\underline{t})$ is given by ℓ as follows

$$\ell \propto r\log(\alpha\,\theta\,\lambda) + \sum_{i=1}^{r} \log t_i^{-2} - (\theta+1)\sum_{i=1}^{r} \log\left(1+\frac{\lambda}{t_i}\right) + (\alpha-1)\sum_{i=1}^{r} \log\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right] + (n-r)\log\left[1-\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^{\alpha}\right], \qquad (18)$$

The derivatives of (18) with respect to α, θ and λ, respectively, are given

$$\frac{\partial \ell}{\partial \alpha} = \frac{r}{\alpha} + \sum_{i=1}^{r} \log\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right] - \frac{(n-r)\left(1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right)\log\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]}{1-\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^{\alpha}}, \qquad (19)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{r}{\theta} - \sum_{i=1}^{r} \log\left(1+\frac{\lambda}{t_i}\right) - (\alpha-1)\sum_{i=1}^{r} \frac{\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\log\left(1+\frac{\lambda}{t_i}\right)}{\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right]} + \frac{(n-r)\alpha\theta\left(1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right)^{\alpha-1}\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\log\left(1+\frac{\lambda}{t_r}\right)}{1-\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^{\alpha}}, \qquad (20)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{r}{\lambda} - (\theta + 1)\sum_{i=1}^{r} \frac{1}{t_i\left(1+\frac{\lambda}{t_i}\right)} - (\alpha - 1)\sum_{i=1}^{r} \frac{\frac{\theta}{t_i}\left(1+\frac{\lambda}{t_i}\right)^{-\theta-1}}{\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right]} - \frac{(n-r)\frac{\alpha\theta}{t_r}\left(1+\frac{\lambda}{t_r}\right)^{-\theta-1}}{\left[1-\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^{\alpha}\right]} \quad (21).$$

Now by setting (19), (20) and (21) to zero and solving them simultaneously, one can get the *Maximum Likelihood Estimate* (MLE) of α, θ and λ which are denoted by $\hat{\alpha}, \hat{\theta}$ *and* $\hat{\lambda}$.

Applying the invariance property the MLEs of the rf and hf are obtained by replacing the parameters α, θ and λ in (3) and (4) by $\hat{\alpha}, \hat{\theta}$ *and* $\hat{\lambda}$.

$$\hat{R}(t_0) = 1 - \left[1 - \left(1 + \frac{\hat{\lambda}}{t_0}\right)^{-\hat{\theta}}\right]^{\hat{\alpha}}, \qquad t_0 > 0, \qquad (22)$$

and

$$\hat{h}(t_0) = \frac{\hat{\alpha}\,\hat{\theta}\,\hat{\lambda}\,t_0^{\,2}\left(1+\frac{\hat{\lambda}}{t_0}\right)^{-\hat{\theta}-1}\left(1-\left(1+\frac{\hat{\lambda}}{t_0}\right)^{-\hat{\theta}}\right)^{\hat{\alpha}-1}}{1-\left[1-\left(1+\frac{\hat{\lambda}}{t_0}\right)^{-\hat{\theta}}\right]^{\hat{\alpha}}}, \qquad t_0 > 0 \qquad (23)$$

Asymptotic variance-covariance matrix

The asymptotic variance-covariance matrix of the estimator's $\hat{\alpha}, \hat{\theta}$ *and* $\hat{\lambda}$ are obtained depending on inverse asymptotic Fisher information matrix. The asymptotic Fisher information matrix can be written as follows:

$$\hat{I}_{ij} = -\left[\frac{\partial^2 \ell}{\partial \varphi_i \partial \varphi_j}\right], i = j = 1,2,3. \qquad (24)$$

where $\varphi_1 = \alpha, \varphi_2 = \theta$ and $\varphi_3 = \lambda$. The elements of the information matrix are derived.

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-r}{\alpha^2} + \left[\frac{(n-r)\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right]^{\alpha+1} \left[log\left(1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right)\right]^2}{\left[1-\left(1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right)^{\alpha}\right]^2} \right], \quad (25)$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = \frac{-r}{\theta^2} - (\alpha - 1)\sum_{i=1}^{r} \left[log\left(1 + \frac{\lambda}{t_i}\right)\right]^2 \left[\frac{\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right] + \left(1+\frac{\lambda}{t_i}\right)^{-2\theta}}{\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right]^2} \right] + \alpha(n-r)\left[log\left(1 + \frac{\lambda}{t_r}\right)\right]^2 \left[\frac{\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right] + \left(1+\frac{\lambda}{t_r}\right)^{-2\theta}}{\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^2} \right], \quad (26)$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = \frac{-r}{\lambda^2} + (\theta + 1)\sum_{i=1}^{r} t_i^{-2}\left(1 + \frac{\lambda}{t_i}\right)^{-2} + \theta\,(\alpha - 1) \sum_{i=1}^{r} \frac{\left[(\theta-1)\left(1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right)\left(1+\frac{\lambda}{t_i}\right)^{-\theta-2} - \theta t_i\left(1+\frac{\lambda}{t_i}\right)^{-2\theta-2}\right]}{t_i^2\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right]^2} - \frac{\alpha\theta(n-r)\left[(\theta-1)\left(1+\frac{\lambda}{t_r}\right)^{-\theta-2}\left[1-\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^{\alpha}\right] - \theta\alpha\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^{\alpha-1}\left(1+\frac{\lambda}{t_r}\right)^{-2\theta-2}\right]}{t_r^2\left[1-\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^{\alpha}\right]^2}. \quad (27)$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial\theta} = -\sum_{i=1}^{r} \frac{\left(1+\frac{\lambda}{t_i}\right)^{-\theta} log\left(1+\frac{\lambda}{t_i}\right)}{\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right]} + \frac{(n-r)\left(1-\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^{\alpha}\right)}{\left[1-\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]^{\alpha}\right]^2}\left[\left(1+\right.\right.$$

$$\left.\frac{\lambda}{t_r}\right)^{-\theta} log\left(1+\frac{\lambda}{t_r}\right) log\left(1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right)\left(1+\left[1-\left(1+\frac{\lambda}{t_r}\right)^{\theta}\right]\right) +$$

$$\left.\left(\left(1+\frac{\lambda}{t_r}\right)^{-\theta} log\left(1+\frac{\lambda}{t_r}\right)\right) log\left(1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right)\right]$$

(28)

$$\frac{\partial^2 \ell}{\partial\alpha\partial\lambda} = \theta\sum_{i=1}^{r} \frac{\left(1+\frac{\lambda}{t_i}\right)^{-\theta-1}}{t_i\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right]} - \frac{(n-r)}{\left[1-\left(1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right)^{\alpha}\right]^2}\left[\frac{-\theta}{t_r}\left(1+\frac{\lambda}{t_r}\right)^{-\theta-1} -\right.$$

$$\frac{\theta}{t_r}\left(1+\frac{\lambda}{t_r}\right)^{-\theta-1}\left[1-\left(1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right)^{\alpha}\right] log\left(1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right) -$$

$$\left.\frac{\theta\alpha}{t_r}\left(1+\frac{\lambda}{t_r}\right)^{-\theta-1} log\left(1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right)\left(1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right)^{\alpha}\right],$$

(29)

$$\frac{\partial^2 \ell}{\partial\theta\partial\lambda} = \sum_{i=1}^{r} \frac{1}{t_i\left(1+\frac{\lambda}{t_i}\right)} - (\alpha - 1)\sum_{i=1}^{r} \frac{t_i\left(1+\frac{\lambda}{t_i}\right)^{-\theta-1}\left(1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right) - \frac{\theta}{t_i}\left(1+\frac{\lambda}{t_i}\right)^{-\theta-1} log\left(1+\frac{\lambda}{t_i}\right)}{\left[1-\left(1+\frac{\lambda}{t_i}\right)^{-\theta}\right]^2} + \frac{\alpha\left(1+\frac{\lambda}{t_r}\right)^{-\theta} log\left(1+\frac{\lambda}{t_r}\right)}{\left[1-\left[1-\left(1+\frac{\lambda}{t_r}\right)^{-\theta}\right]\right]^2}. \quad (30)$$

5. Simulation Study

In this section, a simulation study is presented to illustrate the application of the various theoretical results developed in the previous section

The simulation steps are:-

1- Generated data from IELomax (□, θ, λ) distribution, for different sample sizes (n= 30, 60 and 100).

2- Using number of replications =1000.

3- The computations are performed using R language.

4- For efficiency, we calculate the *relative absolute biases* (RABs), *mean square error* (MSE), variances parameters, (α, θ, λ), RF and HRF as follows:

1- RABs (estimator) = |bias (estimator)|/true value,

2- Variances (estimator) = MSE(estimator) +bias2 ,

Table 1 displays the (RABs), (MSEs) and variances of MLEs and 95% *confidence intervals* (CIs) where the population parameter values are α = 1, θ = 2 and λ =1 based on three levels of Type II censoring 80% and 60% .

Table 1 Relative absolute biases, mean square errors and variances of ML estimates and 95% confidence intervals of the parameters a and b from Kum for diffe rent sample size n, censoring size r ($a = 1$ and $b = 2$)

n	r	estimate	RAB	MSE	variance	SE	LL	UL	length
30	24	0.86342 2.21963	0.13658 0.10981	0.67696 0.96464	0.65831 0.91640	0.82278 0.98216	0.00000 0.25531	2.50898 4.18395	2.50898 3.92864
	18	0.82986 2.27971	0.17004 0.13986	0.70242 0.99832	0.67351 0.92008	0.83811 0.99916	0.00000 0.28139	2.50617 4.27803	2.50617 3.99663
60	48	0.8914 2.0988	0.10860 0.04940	0.34119 0.43714	0.32940 0.42738	0.58412 0.66117	0.00000 0.77647	2.05964 3.42113	2.05964 2.64467
	36	0.8775 2.1978	0.12250 0.09890	0.38324 0.51099	0.36823 0.47187	0.61906 0.71484	0.00000 0.76812	2.11562 3.62748	2.11562 2.85936
100	80	0.9863 2.0384	0.01370 0.01920	0.19783 0.25790	0.19764 0.25643	0.44478 0.50784	0.09674 1.02271	1.87586 3.05409	1.77911 2.03137
	60	0.9678 2.0559	0.03220 0.02795	0.26456 0.34502	0.26352 0.34190	0.51435 0.58739	0.00000 0.88112	1.99650 3.23068	1.99650 2.34955

Table **2** Relative absolute biases, mean square errors and variances of RF and HRF estimates and 95% confidence intervals of RF and HRF from Kum for different sample size n, censoring size r, at (RF = $0.11111, 0.25$ and $0.44444, HRF = 23.75,\ 10.5555\ and\ 5.9375$) at t_0 =0.5, 0.7 and 0.9 respectively.

n	t_0	r	estimator	RAB	MSE	variance	SE	LL	UL	length
30	0.5	24	$\hat{R}$=0.05749 $\hat{h}$=38.96684	1.55044 0.67039	0.25271 231.7107	0.249830 0.158538	0.5027 15.22205	0.0000 8.52274	1.06289 69.41094	1.06289 60.88819
		18	$\hat{R}$=0.04913 $\hat{h}$=43.37635	1.84016 0.86244	0.50165 385.3676	0.49781 0.17381	0.70827 19.63078	0.0000 4.11479	1.46568 82.63792	2.83309 78.52313
	0.7	24	$\hat{R}$=0.15222 $\hat{h}$=14.96517	0.68908 1.50838	0.34118 19.64788	0.33162 0.20318	0.58411 4.43259	0.0000 6.09998	1.32044 23.83036	1.32044 17.73037
		18	$\hat{R}$=0.13506 $\hat{h}$=16.08817	0.81785 1.94049	0.53437 0.23641	0.52116 0.23641	0.73101 5.55394	0.0000 4.98029	1.59707 27.19604	2.92403 22.21575
	0.9	24	$\hat{R}$=0.31314 $\hat{h}$=7.52112	0.38761 2.68157	0.37667 2.76614	0.35943 0.25829	0.61374 1.66317	0.0000 4.19478	1.54061 10.84746	1.54061 6.65269
		18	$\hat{R}$=0.28712 $\hat{h}$=7.87389	0.46004 3.44977	0.63388 0.28173	0.60913 0.28173	0.79617 2.00782	0.0000 3.85826	1.87945 11.88952	3.18467 8.03126
60	0.5	48	$\hat{R}$=0.07905 $\hat{h}$=29.90877	0.91586 0.27404	0.12595 38.01731	0.12492 0.07927	0.35489 6.16582	0.0000 17.57713	0.78883 42.2404	0.78883 24.66327
		36	$\hat{R}$=0.06481 $\hat{h}$=35.57191	1.30912 0.52172	0.25105 139.8444	0.24891 0.08691	0.50105 11.82558	0.0000 11.92074	1.06692 59.22308	1.06692 47.30234
	0.7	48	$\hat{R}$=0.19218 $\hat{h}$=12.50021	0.40705 0.61658	0.16915 3.89989	0.16581 0.10159	0.41128 1.97481	0.0000 8.55058	1.01475 16.44984	1.01475 7.89925
		36	$\hat{R}$=0.16694 $\hat{h}$=14.03477	0.08306 1.17387	0.26748 12.22315	0.26058 0.11821	0.51718 3.49616	0.0000 7.04245	1.20131 21.02709	1.20131 13.98465

	0.9	48	$\hat{R}$=0.36708 $\hat{h}$=6.74181	0.22897 1.09615	0.31055 0.78779	0.17972 0.12914	0.55727 0.88758	0.0000 4.96666	1.48163 8.51696	1.48163 3.55030
		36	$\hat{R}$=0.33438 $\hat{h}$=7.22982	0.32728 2.08688	0.31668 1.81095	0.30457 0.14087	0.56275 1.34572	0.0000 4.53838	1.45987 9.92125	1.45987 5.38286
100	0.5	80	$\hat{R}$=0.0994 $\hat{h}$=26.1952	0.29365 0.1085	0.07509 6.02655	0.07495 0.04756	0.27402 2.4549	0.0000 21.28539	0.64744 31.105	0.64744 9.81961
		60	$\hat{R}$=0.09242 $\hat{h}$=27.63666	0.48928 0.17257	0.14969 15.15828	0.14934 0.05215	0.3869 3.89336	0.0000 19.84993	0.86621 35.42339	1.54759 15.57345
	0.7	80	$\hat{R}$=0.23056 $\hat{h}$=11.3174	0.13051 0.24413	0.09987 0.64135	0.09949 0.06095	0.31602 0.80085	0.0000 9.71571	0.86260 12.91909	0.86260 3.20338
		60	$\hat{R}$=0.21806 $\hat{h}$=11.77092	0.21746 0.38829	0.15737 1.54804	0.15635 0.07093	0.3967 1.2442	0.0000 9.28252	1.01146 14.25932	1.58680 4.97681
	0.9	80	$\hat{R}$=0.42100 $\hat{h}$=6.22256	0.07341 0.43401	0.10838 0.15874	0.10783 0.07748	0.32921 0.39842	0.0000 5.42572	1.07942 7.01939	1.07942 1.59367
		60	$\hat{R}$=0.40462 $\hat{h}$=6.4014	0.12232 0.69029	0.18433 0.29973	0.18274 0.08452	0.42933 0.54747	0.0000 5.30646	1.26329 7.49635	1.71733 2.18989

Table 3 Relative absolute biases, mean square errors and variances of ML estimates and 95% confidence intervals of the parameters a and b from Kum for different sample size n, censoring size r ($a = 10$ and $b = 5$)

n	r	estimate	RAB	MSE	variance	SE	LL	UL	length
30	24	9.6183 4.7582	0.03817 0.04836	0.31029 0.35410	0.16460 0.29563	0.55704 0.59506	8.50422 3.56808	10.73238 5.94832	2.22816 2.38024
	18	9.44970 4.59968	0.05503 0.08006	0.48368 0.50206	0.18085 0.34180	0.69547 0.70856	8.05876 3.18256	10.84064 6.01680	2.78188 2.83424
60	48	9.8832 4.8672	0.01168 0.02656	0.09594 0.16546	0.08230 0.14782	0.30975 0.40676	9.26371 4.05367 1.25068	10.50269 5.68073 3.21952	1.23898 1.62705 1.96885
	36	9.8164 4.8648	0.01836 0.02704	0.12414 0.18918	0.09043 0.17090	0.35233 0.43495	9.11173 3.99491 1.20859	10.52107 5.73469 3.55461	1.40933 1.73979 2.34602
100	80	9.8846 4.97604	0.01154 0.00479	0.05401 0.07748	0.04069 0.07691	0.23239 0.27836	9.41981 4.41932	10.34939 5.53276	0.92958 1.11344
	60	9.93183 4.95802	0.00682 0.00840	0.05891 0.08818	0.05426 0.08642	0.24271 0.29696	9.44641 4.36411	10.41725 5.55193	0.97083 1.18782

Table **4** Relative absolute biases, relative mean square errors and variances of RF and HRF estimates and 95% confidence intervals of rf and hf from Kum for different sample size n, censoring size r, RF=0.0032, 0.01165and 0.0404, $HRF = 47.9478,\ 13.04908\ and\ 3.66489$, at t0 $=$ 0.5, 0.7 and 1 respectively.

n	t_0	r	estimator	RAB	MSE	variance	SE	LL	UL	length
30	0.5	24	$\hat{R}$=0.00193	5.12128	0.19493	0.19493	0.44151	0.0000	0.88495	0.88495
			$\hat{h}$=49.15845	0.02671	1.51782	0.05215	1.23200	46.69446	51.62244	4.92799
		18	$\hat{R}$=0.00217	4.64576	0.25341	0.25341	0.5034	2.0136	1.00897	2.0136
			$\hat{h}$=45.16668	0.05825	7.80244	0.06779	2.79329	39.5801	50.7533	11.1731
	0.7	24	$\hat{R}$=0.00701	1.40449	0.11566	0.11563	0.34008	0.0000	0.68718	0.68718
			$\hat{h}$=13.45782	0.09814	0.23799	0.07093	0.48784	12.48213	14.43351	1.95138
		18	$\hat{R}$=0.00764	1.27408	0.15035	0.15033	0.38774	1.5509	0.78313	1.55098
			$\hat{h}$=12.79134	0.21404	0.15864	0.09221	0.39830	11.9947	13.5879	1.59320
	0.9	24	$\hat{R}$=0.02476	0.40508	0.08298	0.08274	0.28807	0.0000	0.6009	0.6009
			$\hat{h}$=3.74956	0.34942	0.09169	0.08452	0.3028	3.14396	4.35516	1.21121
		18	$\hat{R}$=0.02614	0.36746	0.10776	0.10756	0.32827	1.31309	0.68269	1.31309
			$\hat{h}$=3.67368	0.76211	0.10996	0.10988	0.3316	3.01048	4.33688	1.32639
60	0.5	48	$\hat{R}$=0.00253	2.71034	0.09746	0.09746	0.31219	0.0000	0.62690	0.62690
			$\hat{h}$=48.42812	0.01078	0.25677	0.02607	0.50673	47.41466	49.4415	2.0269
		36	$\hat{R}$=0.00195	4.76868	0.12671	0.12671	0.35597	0.0000	0.71389	0.71389
			$\hat{h}$=51.07598	0.06687	9.81938	0.03389	3.13359	44.8088	57.34316	12.5343
	0.7	48	$\hat{R}$=0.00919	0.74330	0.05783	0.05782	0.24047	0.0000	0.49013	0.49013

			$\hat{h}$=13.23525	0.03960	0.07012	0.03546	0.26480	12.70565	13.76486	1.0592
		36	$\hat{R}$=0.00723 $\hat{h}$=13.7426	1.30779 0.24569	0.07518 0.52718	0.07516 0.04611	0.27419 0.72607	0.0000 12.29054	0.55561 15.19481	0.55561 2.90428
	0.9	48	$\hat{R}$=0.03212 $\hat{h}$=3.70609	0.21438 0.14101	0.04144 0.04396	0.04137 0.04226	0.20356 0.20966	0.0000 3.28677	0.43925 4.12541	0.43925 0.83864
		36	$\hat{R}$=0.02587 $\hat{h}$=3.77634	0.37719 0.87481	0.05399 0.56182	0.05378 0.54940	0.23236 0.74955	0.0000 2.27725	0.49059 5.27544	0.49059 2.99819
100	0.5	80	$\hat{R}$=0.00282 $\hat{h}$=48.9063	1.40238 0.02044	0.58476 0.93452	0.58476 0.01564	0.58476 0.93452	0.0000 46.97297	1.53221 50.8398	3.05878 3.86683
		60	$\hat{R}$=0.00265 $\hat{h}$=49.3765	2.02873 0.03047	0.07602 2.06172	0.07602 0.02033	0.27572 1.43587	0.0000 46.5048	0.55408 52.2483	0.55408 5.74348
	0.7	80	$\hat{R}$=0.01032 $\hat{h}$=13.2495	0.38460 0.07509	0.02482 0.06145	0.02482 0.02128	0.02482 0.06145	0.0000 12.75373	0.32542 13.7452	0.63020 0.99154
		60	$\hat{R}$=0.00973 $\hat{h}$=13.3514	0.55637 0.11198	0.04509 0.11910	0.04509 0.02767	0.21235 0.34511	0.0000 12.6612	0.43444 14.0416	0.43444 1.38044
	0.9	80	$\hat{R}$=0.03614 $\hat{h}$=3.69668	0.11092 0.26735	0.02484 0.02637	0.02482 0.02536	0.02484 0.02637	0.0000 3.37190	0.35134 4.02146	0.63041 0.64956
		60	$\hat{R}$=0.03423 $\hat{h}$=3.71259	0.16047 0.3987	0.03231 0.33192	0.03227 0.32964	0.17974 0.57612	0.0000 2.56035	0.39372 4.86483	0.39372 2.30448

6. Application

In this section, the application of real data sets is provided to illustrate the importance of the IELomax distribution. To check the validity of the fitted model, Kolmogorov-Smirnov (KS) goodness of fit test is performed for data set and the p $= 6.16(10)^{-6}$ values in each case indicates that the model fits the data very well. The KS statistics compares the empirical cumulative distribution of the data to any specified continuous distribution when its parameters are estimated by maximum likelihood. This comparisons of the two CDF looks only at point of maximum discrepancy of this statistics. The lower are the KS values more evidence that the specified model generates the data. More details of this test are described by Evans et al.[12]. All computations are carried out using the R-software. The data set is the failure times of 84 Aircraft Windshield. The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the nonstructural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the windshield. We consider the data on failure times for a particular model windshield given in Murthy et al.[13]. These data were recently studied by Ramos et al.[14] and El-Bassiouny et al.[8].

Table 5: The inverse failure times of 84 Aircraft Windshield

0.5359	0.4193	0.2904	3.3223	0.5330	0.4031	0.2884	3.2362	0.2145
0.5266	0.3831	0.2875	1.7953	0.5233	0.3810	0.2795	1.0604	0.5230
0.3799	0.2782	0.9346	0.5225	0.3779	0.2703	0.8897	0.5048	0.3758
0.2646	0.8013	0.4975	0.3720	0.2548	0.7806	0.4907	0.3546	0.3333
0.2478	0.7806	0.4796	0.3460	0.2427	0.7675	0.4787	0.3446	0.2400
0.6983	0.4769	0.3408	0.2358	0.6757	0.4684	0.3376	0.2350	0.6645
0.4643	0.3374	0.2338	0.6640	0.4566	0.3333	0.2323	0.6378	0.4558
0.3223	0.2285	0.6192	0.4498	0.3211	0.2248	0.6177	0.4496	0.3208
0.2230	0.6053	0.4486	0.3159	0.2188	0.6053	0.4348	0.2990	0.2173
0.5692	0.4303	0.2962						

It is clear from, Tables 6, RABs, variances and MSE of the MLEs of the shape parameters α, θ and λ decrease with a big sample size. Also, it is observed that as the level of censoring decreases the RABs, variances and RMSEs of the estimates decrease. The length of the confidence interval becomes narrower as the level of censoring decreases. Table 6 indicate that RF increases and HRF decreases when the mission time t_0 increases, also RABs and MSEs of M LEs of RF and HRF decrease as the level of censoring decreases.

Table 6: Relative absolute biases, relative mean square errors and variances of ML estimates and 95% confidence intervals of the parameters α, θ and λ from IELomax for a real data, with different censoring size r and repetitions

n	r	Estimate	RAB	MSE	Variance	SE	LL	UL	length
	50	$\hat{\alpha}$ = 0.90916	0.09084	0.04131	0.03306	0.20325	0.50265	1.31567	0.81301
		$\hat{\theta}$= 1.94929	0.02536	0.10104	0.09847	0.31787	1.31355	2.58503	1.27148
		$\hat{\lambda}$ = 1.10815	0.10815	0.15145	0.13975	0.38916	0.32983	1.88647	1.55664
84	60	$\hat{\alpha}$ = 0.93296	0.06704	0.03068	0.02619	0.17517	0.58262	1.2833	0.70068
		$\hat{\theta}$= 1.96971	0.01519	0.07751	0.07659	0.2784	1.41291	2.52651	1.11361
		$\hat{\lambda}$ = 1.03015	0.03015	0.09103	0.09012	0.30171	0.42673	1.63357	1.20683
	80	$\hat{\alpha}$ = 0.98914	0.01086	0.01282	0.0127	0.11322	0.76271	1.21557	0.45287
		$\hat{\theta}$= 1.98908	0.00546	0.04873	0.04861	0.22075	1.54759	2.43057	0.88299
		$\hat{\lambda}$ = 1.01817	0.01817	0.0628	0.06247	0.2506	0.51697	1.51937	1.0024

Table 7: Relative absolute biases, relative mean square errors and variances of RF and HRF estimates and 5% confidence intervals of RF and HRF from IELomax for a real data, with different censoring size r (RF = 0.11111,0.16955 and 0.2500, and HRF = 3.03243,1.98724 and 1.34775.

n	r	t_0	Estimate	RAB	MSE	Variance	SE	LL	UL	length
		0.5	$\hat{R}$ = 0.0937	0.42049	0.02000	0.0197	0.14142	0.000	0.37654	0.37654
			$\hat{h}$ = 3.03243	0.04654	0.03903	0.02607	0.19757	2.75115	3.54143	3.54143
	50	0.7	$\hat{R}$ = 0.14406	0.27556	0.01989	0.01924	0.14142	0.0000	0.42614	0.42614
			$\hat{h}$ = 1.9872	0.07101	0.04049	0.03546	0.20123	2.75115	2.46063	2.4603
		0.9	$\hat{R}$ = 0.21493	0.18688	0.11404	0.11284	0.3377	0.0000	0.89032	0.89032
			$\hat{h}$ = 1.34775	0.1047	0.04418	0.04226	0.21018	0.97116	1.8119	1.8119
		0.5	$\hat{R}$ = 0.10345	0.19517	0.00702	0.00696	0.08378	0.0000	0.27102	0.27102
			$\hat{h}$ = 3.03367	0.00503	0.02607	0.02607	0.16147	2.71074	3.35661	0.64587
84	60	0.7	$\hat{R}$ = 0.15791	0.1279	0.00598	0.00585	0.07733	0.00324	0.31257	0.30934
			$\hat{h}$ = 1.99640	0.00767	0.03554	0.03546	0.18853	1.61934	2.37346	0.75412
		0.9	$\hat{R}$ = 0.23339	0.08675	0.01276	0.01248	0.11295	0.00749	0.45929	0.4518
			$\hat{h}$ = 1.35987	0.01131	0.04241	0.04226	0.20593	0.94801	1.77173	0.82372
		0.5	$\hat{R}$ = 0.10867	0.05906	0.00058	0.00057	0.024	0.06067	0.15667	0.0960
			$\hat{h}$ = 3.04728	0.00612	0.02629	0.02607	0.16214	2.72299	3.37156	0.64857
	80	0.7	$\hat{R}$ = 0.16596	0.03871	0.00238	0.00237	0.04881	0.06823	0.26359	0.19526
			$\hat{h}$ = 1.99674	0.00934	0.03555	0.03546	0.18855	1.61965	2.37384	0.75419
		0.9	$\hat{R}$ = 0.24508	0.02625	0.00962	0.0096	0.0981	0.04888	0.44129	0.39241
			$\hat{h}$ = 1.35359	0.01378	0.04229	0.04226	0.20566	0.94228	1.7649	0.82262

7. Concluding Remarks

From Tables 1 and 3, one can observe that the RABs, variances and MSE of the MLEs of the shape parameters α , θ and λ decrease when the sample size n increases. Also, it is observed that as the level of censoring decreases the RABs, variances and MSEs of the estimates decrease. The length of the confidence interval becomes narrower as the sample size increases.

- Table 2 and 4 indicate that the RF decreases and the HRF increases when the mission time t_0 increases, also the RABs and MSEs of the MLEs of the RF and the HRF decrease when the sample size increases.
- These results are expected since decreasing the level of censoring means that more information is provided by the sample and hence increase the accuracy of the estimates.

In general, when r = n, all the results obtained for Type II censored sample reduce to those of the complete sample

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