Kumaraswamy-generalized distribution as a general class

Prof. Tarek Mohamed shams Eldin Hegap

Department of Business Administration – College of Business - King Khalid University
Kingdom of Saudi Arabia
tshams2000@yahoo.com

Kumaraswamy-generalized distribution as a general class

Prof. Tarek Mohamed shams Eldin Hegap

Department of Business Administration – College of Business - King Khalid University

Kingdom of Saudi Arabia

tshams2000@yahoo.com

Abstract

This article introduces the review of literature for Kumaraswamy-generalized distribution as a general class; we review the most fields which covered for Kumaraswamy-generalized distribution. The new general class distribution can have a decreasing and upside-down bathtub failure rate function depending on the value of its parameters; it's including the Kumaraswamy-generalized Gamma distribution, some special sub-model depending on the original form. The Kumaraswamy-generalized Pareto distribution. On generalized order statistics from Kumaraswamy distribution. Some structural properties of the proposed distribution are reviewed, including explicit expressions for the moments, order statistics and their moments. The most methods of estimation are covered.

Keywords: Kumaraswamy-generalized distribution, Kumaraswamy-generalized Gamma distribution, Kumaraswamy-generalized Pareto distribution, maximum likelihood estimation, hazard rate function, order statistics. Moment generating function.

1 - Introduction

The theory of compound distributions is well known and frequently used in various scientific disciplines. In particular, it has useful applications in industrial reliability and medical survivorship analysis. The modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields. In this proposal, we introduce the new general class of compound distributions which supported us with new families of distributions and sub-models. The new distributions can have a decreasing and upside-down bathtub failure rate function depending on the values of its parameters.

For the new general class, we present the family of Kumaraswamy generalized (denoted with the prefix "Kw-G" for short) distributions introduced by Cordeiro and de Castro (2011)[11]. Nadarajah et al. (2011)[56] studied some mathematical properties of this family. The Kumaraswamy (Kw-G) distribution is not very common among statisticians and has been little explored in the literature. It does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution been widely appreciated. However, in a very recent paper, Jones (2009)[34] explored the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and Kw distributions. The Kumaraswamy Generalized (Kw-G) distribution, which stems from the following general construction, if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F(t;a,b) = 1 \quad (1 \quad G(t)^a)^b \tag{1.1}$$

where $a > 0$ and $b > 0$ are two additional shape parameters which govern skewness and tail weights, the Kw-G distribution can be used quite effectively even if the data are censored. Correspondingly, its density function is distributions has a very simple form

$$f(t;a,b) = ab\, g(t)\, G(t)^{a-1}(1 - G(t)^a)^{b-1} \tag{1.2}$$

The density family (1.2) has many of the same properties of the class of beta-G distributions (see Eugene and Famoye (2002))[21], but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta-G family of distributions, special Kw-G distributions can be generated as follows: the Kw-normal distribution is obtained by taking G(t) in (1.1) to be the normal cumulative function. Analogously, the Kw-Weibull (Cordeiro et al. (2010)[12]), Kw-generalized gamma (Pascoa et al. [2011][50]), and Kw-Gumbel (Cordeiro et al. (2012)[55].) distributions are obtained by taking G(t) to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, among several others. Hence, each new Kw-G distribution can be generated from a specified G distribution. Nadarajah and Eljabri (2013) [57] have proposed the mathematical properties of the GP distribution. Espa (2009)[5] obtain, the joint distribution, distribution of product and distribution of ratio of two generalized order statistics from Kumaraswamy distribution.

1.1 Review of Literature

Kumaraswamy (1980)[38] introduced his now eponymous distribution, originally called double-bounded distribution, as an alternative to the beta distribution. They have the same real parameters, $(a,b) > 0$ the same support and similar shapes, but the Kumaraswamy distribution function, unlike the beta distribution function, has a closed algebraic form. Is has been found both more accurately fitting hydrological data in simulations [Kumaraswamy, 1980[38]] and computationally more tractable [Jones, 2009][34]. The similarity between the two classes can be formalized. It is known and easy to see that, if random variable T is Kumaraswamy-distributed with parameters a and b, then T^a is beta distributed with $a = 1$ and same b.

The probability density function (pdf) and its cumulative (cdf) respectively

$$f(t) = ab\, t^{a-1}(1-t^a)^{b-1} \qquad 0 < t < 1 \;, (a,b) > 0 \tag{1.3}$$

$$F(t) = 1-(1-t^a)^b \qquad 0 < t < 1 \;, (a,b) > 0 \tag{1.4}$$

In hydrology and related areas, the Kum distribution has received considerable interest, see Sundar and Subbiah (1989)[68], Fletcher and Ponnambalam (1996)[23], Seifi et al. (2000)[64] and Ganji et al. (2006)[24]. According to Nadarajah (2008)[54], many papers In the hydrological literature have used this distribution because it is demand as a "better alternative " to the beta distribution, see, for example, Koutsoyiannis and Xanthopoulos (1989)[37].

Jones (2008)[33] illustrated that the kumaraswamy distribution has some properties like the beta distribution such as both densities are unimodal, uniantimodal, increasing, and decreasing or constant depending on the values of its parameters. Also, Jones (2008)[33] highlighted several advantages of Kw distribution over beta distribution, the Kw distribution much simpler to use especially simulation studies due to the simple closed form of both its cumulative distribution function and quantile function, the normalizing constant is very simple, simple explicit formula for the distribution and quantile function which do not involve any special functions, a simple formula for random variable generation, Explicit formula for L-moments and simpler for moments of order statistics. Further, he mentioned that the beta distribution has the following advantages over th Kw distribution, simpler formula for moments and moment generating function; a one parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution via physical processes.

The family of Kumaraswamy generalized (denoted with the prefix “Kw-G” for short) distributions introduced by Cordeiro and de Castro (2011)[11]. Nadarajah et al. (2011)[56]studied some mathematical properties of this family. The Kumaraswamy (Kw) distribution is not very common among statisticians and has been little explored in the literature. However, in a very recent paper, Jones (2009)[34] explored the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and Kw distributions. The Kumaraswamy Generalized (Kw-G) distribution, which stems from the following general construction, if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F(t; a, b) = 1 - (1 - G(t)^a)^b \tag{1.5}$$

where $a > 0$ and $b > 0$ are two additional shape parameters which govern Skewness and tail weights, the Kw-G distribution can be used quite effectively even if the data are censored. Correspondingly, its density function is distributions has a very simple form

$$f(t; a, b) = ab\, g(t)\, G(t)^{a-1}(1 - G(t)^a)^{b-1} \tag{1.6}$$

The density family (1.4) has many of the same properties of the class of beta-G distributions (see Eugene and Famoye (2002)[21]), but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta-G family of distributions, special Kw-G distributions can be generated as follows: the Kw-normal distribution is obtained by taking G(t) in (1.5) to be the normal cumulative function. Analogously, the Kw-Weibull (Cordeiro et al. (2010)[12]), Kw-generalized gamma (Pascoa et al. [2011][50]), and Kw-Gumbel (Cordeiro et al. (2011)[11].) distributions are obtained by taking G(x) to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, among several others. Hence, each new Kw-G distribution can be generated from a specified G distribution. Nadarajah and Eljabri (2013)[57] have proposed the mathematical properties of the GP distribution. Espa (2009)[5] obtain, the joint distribution, distribution of product and distribution of ratio of two generalized order statistics from Kumaraswamy distribution.

1.2 The Reliability Function

According to Meeker and Escobar (1998)[51] the stress-strength model describe the life of a component which has a random strength X_1 that is subjected to a random stress X_2

The component fails at the instant that the stress applied to it exceeds strength, and the component will function satisfactorily whenever $X_1 > X_2$, so $R = \Pr(X_2 < X_1)$ is a measure of component reliability.

The definition of reliability, when X_1 and X_2 are independent random variables belonging to the same univariate family of distributions, is

$$R = \int_{-\infty}^{\infty} f_1(x).F_2(x)dx$$

Where

$f_1(x)$: is the pdf of for the random variable X_1.

$F_2(x)$: is the cdf of distribution the random variable X_2.

Reliability has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the aging of concrete pressure vessels, in the area of stress-strength models.

1.2.1 Hazard Function of the class of Kw-G Distributions

Cordeiro et al. (2010a)[12] obtained Hazard function of the class of Kw-G distributions where the survival function is expressed by

$S(x) = 1 - F(x)$ substituting from equation (1.5) into the last equation, yields:

$$S(x) = (1 - G(t)^a)^b \qquad (1.7)$$

Furthermore, the hazard function is expressed by

$$H(x) = \frac{f(x)}{S(x)} \qquad (1.8)$$

Substituting from (1.6) and (1.7) into the last equation, yields:

$$H(x)=\frac{ab\,g(t)G(t)^{a-1}(1-G(t)^{a})^{b-1}}{(1-G(t)^{a})^{b}} \qquad (1.9)$$

then

$$H(x)=\frac{ab\,g(t)G(t)^{a-1}}{(1-G(t)^{a})} \qquad (1.\ 10)$$

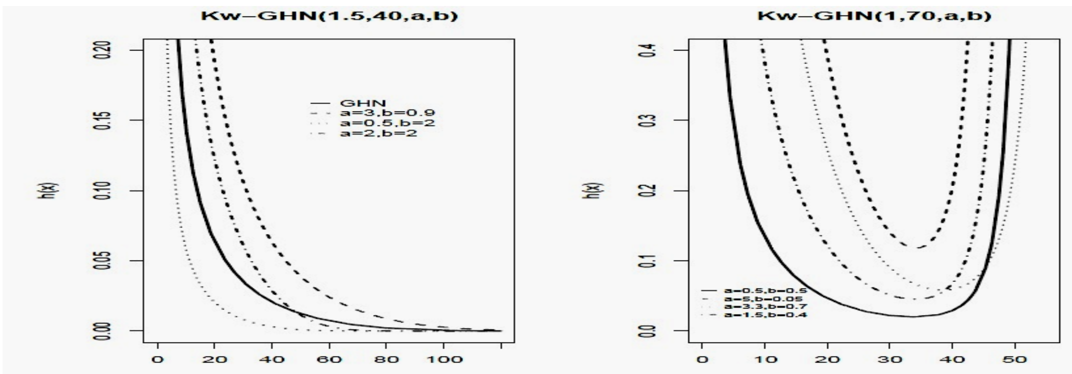


Figure 1: plots of hazard function for The Kumaraswamy Generalized Half-Normal Distributio

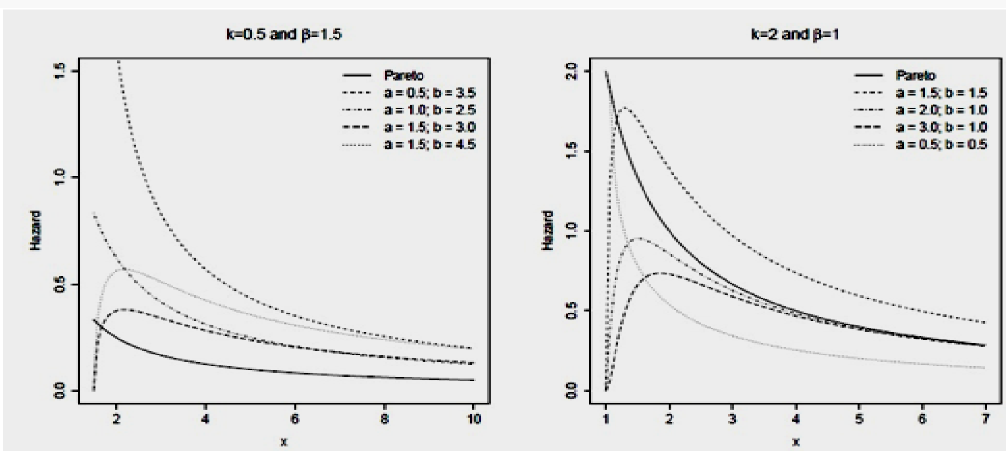


Figure 2: plots of The Kumaraswamy Pareto hazard function for som parameter value

1-3 A General Expansions For The Density Function

Cordeiro et al. (2010 a)[12] derived an expansion from equation (1.2) as follows: For b>0 real non-integer, the series representation be used

$$(1-G(t)^a)^{b-1} = \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} G(t)^{ai}$$

(1.11)

Where the binomial coefficient is defined for any real, from the above expansions and formula (1.2) the density function of the class of Kw-G distributions van be written as

$$f(t) = g(t)\sum_{i=0}^{\infty} w_i G(t)^{a(i+1)-1}$$

Where,

$$w_i = w_i(a,b) = (-1)^i \binom{b-1}{i} ab \quad and \quad \sum_{i=0}^{\infty} w_i = 1$$

1-4 General Formula for moments

Cordeiro et al. (2010)[12] expressed the s^th moments for the class of Kw-G distributions as an infinite weighted sum of Probability weighted moments of order (s,i) for the parent distribution G from equation (2) for a integer and from (1.4) , (1.5) for a real non-integer. We assume Y and X following the baseline G and the class of Kw-G distributions, respectively. The s^th moment of X, say μ'_S , can be expressed in terms of the (s,k)^th Probability weighted moments, as:

$\tau_{r,s} = E\{t^r G(t)^s\}$ Of Y for k=0, 1, … (1.12)

General formula for moments. If a is integer: Cordeiro et al. (2010)[12] Derived general formula for moments when a is integer from the fact that

$$\mu'_r = \int_{-\infty}^{\infty} t^r f(t)\,dt$$

Substituting from (4) into μ'_S, yields

$$\mu'_r = \int_{-\infty}^{\infty} t^r g(t) \sum_{i=0}^{\infty} w_i G(t)^{a(i+1)-1}\,dt \qquad (1.13)$$

From probability weighted moment's definition:

$$\mu_r' = \sum_{i=0}^{\infty} w_i \tau_{r,a(i+1)-1} \tag{1.14}$$

Hence

$$\tau_{r,a(i+1)-1} = \int_{-\infty}^{\infty} t^r g(t) G(t)^{a(i+1)-1} dt$$

1-5 Order Statistics of the class of Kw-G distributions:

Moments of order statistics play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. We now derive an explicit expression for the density function of the ith order statistic $t_{i:n}$, say $f_{i:n}(t)$, in a random sample of size n from the Kw-GP distribution. We can write

$$f_{i:n}(t) = \frac{n!}{(i-1)!(n-i)!} f(t) F^{i-1}(t)[1-F(t)]^{n-i}$$

(1.15)

Where $(.)\ and\ F(.)$, are the pdf and cdf of the Kw-GP distribution, by using binomial expansion of $[1-F(t)]^{n-i}$, yields

$$f_{i:n}(t) = \frac{n!}{(i-1)!(n-i)!} f(t) \sum_{i=0}^{n-i} (-1)^i \binom{n-i}{i} F(t)^{n+i-1}$$

(1.16)

An expansion for the density of order statistics of the class of Kw-G distributions is presented as a function of baseline density multiplied by infinite weighted sums of powers for G(t). This results enables us to derive ordinary moments of order statistics of the class of Kw-G Distribution as infinite weighted sums of probability weighted moments for the G distribution.

2- Some Special Distributions of Kumaraswamy Distributions

2.1 The Kumaraswamy-Generalized Gamma Distribution

The Kumaraswamy-Generalized Gamma(Kum-GG) Distribution was introduced by Marcelino A. R. Pascoa et al (2011)[50]; they are discussed in their paper an expression for the sth moment, properties of the hazard function, and results for the distribution of the sum of Kum-GG random variables, maximum likelihood estimation and some asymptotic results.

be the cdf of the GG distribution (Stacy, 1962) [66] given by $G(t;\alpha,\tau,k)$ Let

$$G(t;\alpha,\tau,k)=\frac{\gamma[k,(t/\alpha)^{\tau}]}{\Gamma(k)}$$

is the incomplete gamma function $\gamma(k,x)=\int_0^x \omega^{k-1}\,e^{-\omega}\,d\omega$ Where α > 0, τ > 0 , k > 0 ,

and $\Gamma(.)$ is the gamma function . Basic properties of the GG distribution are given by Stacy and Mihran (1967)[66] and Lawless (1980[40] , 2003[41]). Some important distributions that are special sub-models of the GG distribution in the following table

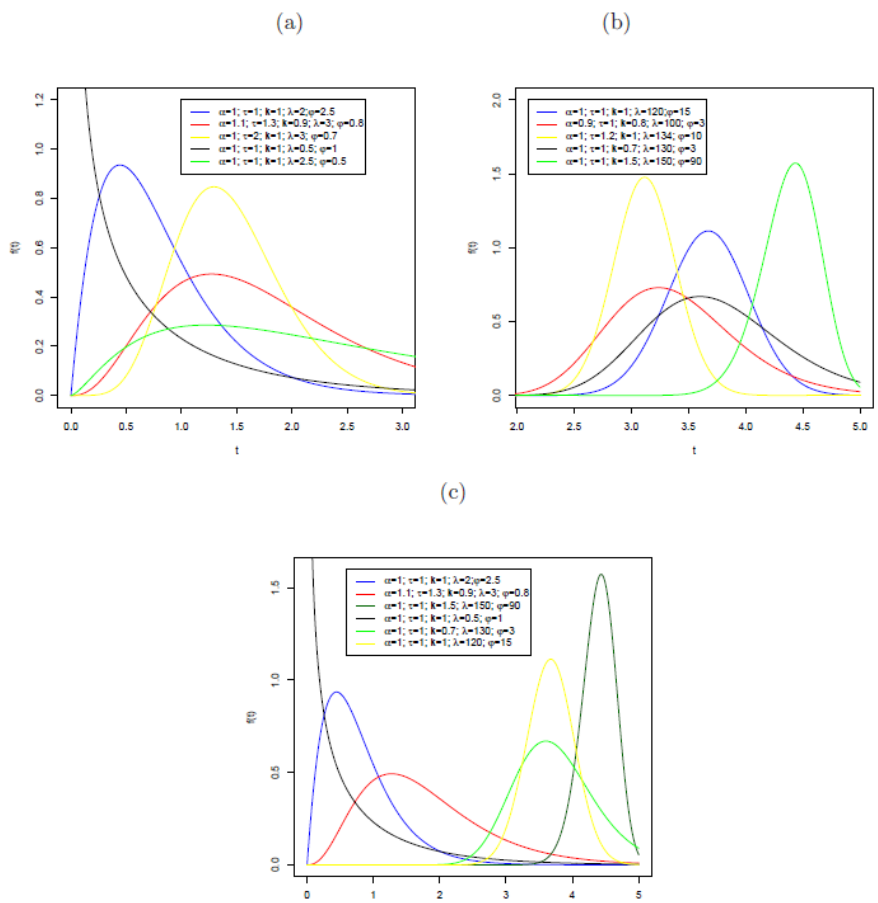
Distribution	τ	α	k
Gamma	1	α	k
Chi-square	1	2	$n/2$
Exponential	1	α	1
Weibull	C	α	1
Rayleigh	2	α	1
Maxwell	2	α	$3/2$
Folded normal	2	$\sqrt{2}$	$1/2$

The cdf of the Kum GG distribution can be defined by substituting $G(t;\alpha,\tau,k)$ into equation (1.5). Hence, The density function of the Kum-GG distribution (Cordeiro and Castro, 2011)[11] is given by

$$f(t)=\frac{ab\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}exp\left\{-\left(\frac{t}{\alpha}\right)^{\tau}\right\}\left\{\gamma_1\left(k,\left(\frac{t}{\alpha}\right)^{\tau}\right)\right\}^{a-1}\left(1-\left\{\gamma_1\left(k,\left(\frac{t}{\alpha}\right)^{\tau}\right)\right\}^{a}\right)^{b-1}$$

(2.1)

Where $t>0$ and $(\alpha,\tau,a,b,k)>0$, Here, $\gamma_1(.,.)$ is the incomplete gamma ratio function defined by, $\gamma_1(k,x)=\frac{\gamma(k,x)}{\Gamma(k)}$ i.e. the cdf of the standard gamma distribution with parameter k.



igure 3 :plots of the density function of the Ku-GG distribution with different shape values

If T is a random variable with density function (2.1), we write T~ KumGG (α,τ,k,a,b). The survival and hazard rate functions corresponding to (2.1) are

$$(2.2) \quad S(t)=\left(1-\left[\frac{1}{\alpha^{\tau k}\Gamma(k)}\sum_{j=0}^{\infty}\frac{(-1)^{j}\alpha^{-(\tau j)}t^{\tau(k+j)}}{(k+j)j!}\right]^{a}\right)^{b}$$

and

$$h(t)=\frac{\mathrm{ab}\tau}{\alpha\Gamma(\mathrm{k})}\left(\frac{t}{\alpha}\right)^{\tau k-1} exb\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\left\{\gamma_{1}\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{a-1}\left(1\text{-}\left\{\gamma_{1}\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{a}\right)^{-1} \quad (2.3)$$

respectively. Plots of the KumGG density and survival rate function for selected parameter values are given i n following figures (4)

(a) (b)

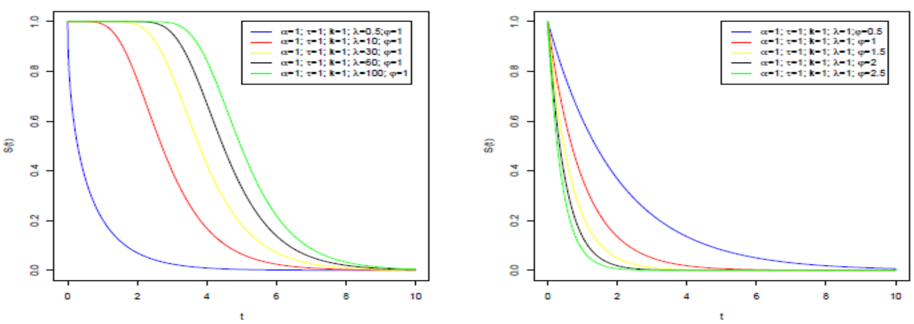


Figure 4: (a) plots of the the Ku-GG survival function for some value of a.

(b) Plots of the the Ku-GG survival function for some value of b.

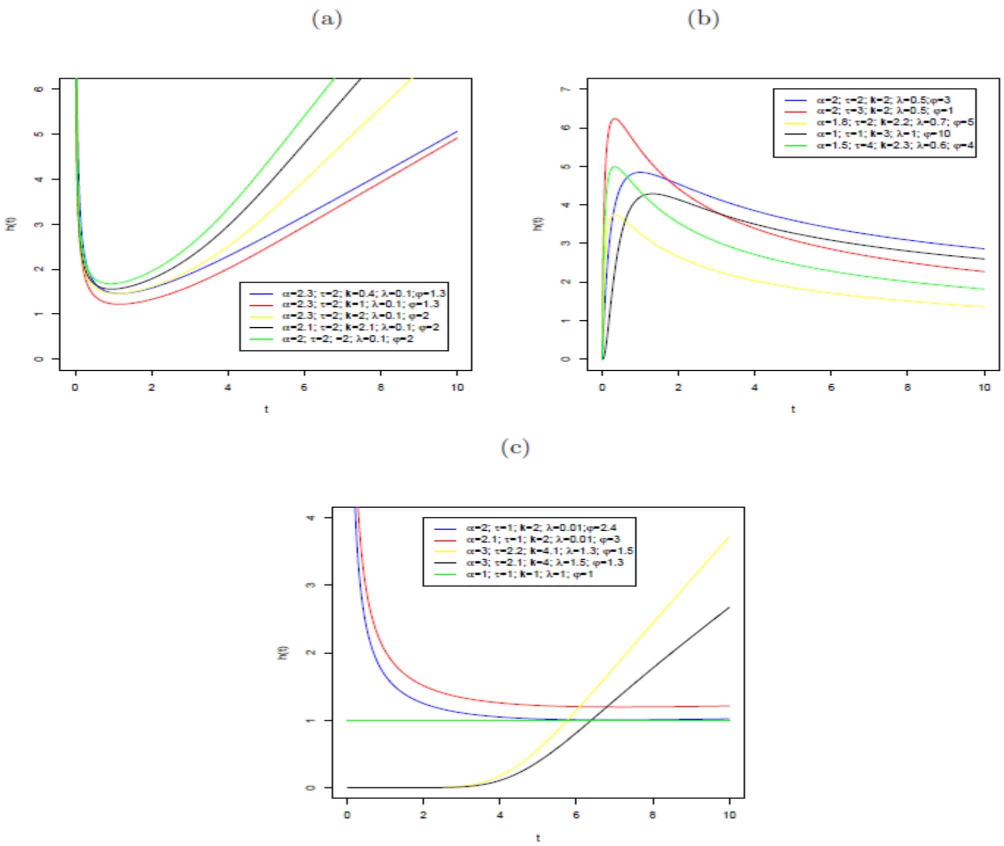


Figure 5: KumGG hazard rate function.

(a) The distribution has a bathtub hazard rate function.

(b) The distribution has an unimodal hazard rate function.

(c) The distribution has increasing, decreasing hazard rate function.

2.2 Special Distribution

The following well-known and new distributions are special sub-models of the KumGG distribution.

2.2.1 Exponantiated Generalized Gamma distribution

If $b=1$, the KumGG distribution reduces to

$$f(t)=\frac{a\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{a-1},\ t>0, \tag{2.4}$$

Which is the exponentiated generalized gamma (EGG) density introduced by Cordeiro et al.(2009)[13]. If τ=b=1 in addition to k=1, the special cases corresponds to the exponentiated exponential (EE) distribution proposed by Gupta and Kundu (1999[27], 2001[26]). If τ=2 in addition to k=b=1, the special case becomes the generalized Rayleigh (GR) distribution (Kundu and Raqab, 2005) [39].

2.2.2 Kum-Weibull distribution (Cordeiro and Castro, 2010)

For k=1, equation (2.1) yields

$$f(t)=\frac{ab\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau-1}exp\left\{-\left(\frac{t}{\alpha}\right)^{\tau}\right\}\left\{1-exp\left(-\left(\frac{t}{\alpha}\right)^{\tau}\right)\right\}^{a-1}\left(1-\left\{1-exp\left(-\left(\frac{t}{\alpha}\right)^{\tau}\right)\right\}^{a}\right)^{b-1},\quad t>0 \tag{2.5}$$

Which is the Kum- Weibull (KumW) distribution If b = k = 1, it reduces to the exponentiated Weibull (EW) distribution (see, Mudholkar et al., 1995[52], 1996[53]). If b = a = k =1, (2.1) becomes the Weibull distribution. If τ = 2 and k = 1, we obtain the Kum-Rayleigh (KumR) distribution. If k = τ = 1, we obtain the Kum-exponential (KumE) distribution. If b = λ = k =1 we obtain two important special sub- models: the exponential (τ=1) and Rayleigh (τ=2) distributions, respectively.

2.2.3 Kum-Gamma distribution (Cordeiro and Castro, 2010)

For τ= 1, the KumGG distribution reduces to

$$f(t)=\frac{ab}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{k-1}exp\left\{-\left(\frac{t}{\alpha}\right)\right\}\left\{\gamma_1\left(k,\left(\frac{t}{\alpha}\right)\right)\right\}^{a-1}\left(1-\left\{\gamma_1\left(k,\left(\frac{t}{\alpha}\right)\right)\right\}^{a}\right)^{b-1},t>0 \tag{2.6}$$

Which is the four parameter Kum- Gamma (KumG4) distribution. If $b=\tau=1$ we obtain the exponentiated gamma (EG3) distribution with three parameters. If $b = \tau = \alpha = 1$, the special case corresponds to the exponentiated gamma (EG2) distribution with two parameters. Further, if $k = 1$, we obtain the Kum- Gamma distribution with one parameter. If we take $b=a=\tau=1$, the special case corresponds to the two parameter gamma distribution. In addition, if $k = 1$, we obtain the one parameter gamma distribution.

2.2.4 Kum-Chi-Square distribution

If we take $\tau = 2$, $\alpha = 2$ and $k = \frac{p}{2}$, the density is given by

$$f(t)=\frac{ab}{2\Gamma\left(\frac{p}{2}\right)}\left(\frac{t}{2}\right)^{\frac{p}{2}-1}\exp\left[-\left(\frac{t}{2}\right)\right]\left\{\gamma_1\left[\frac{p}{2},\left(\frac{t}{2}\right)\right]\right\}^{a-1}\left(1-\left\{\gamma_1\left[\frac{p}{2},\left(\frac{t}{2}\right)\right]\right\}^{a}\right)^{b-1}, \quad t>0 \tag{2.7}$$

Which is the Kum-Chi-Square (Kum-Chi) distribution, If $b=1$, $\tau = \alpha = 2$ and $k = \frac{p}{2}$, we obtain the exponentiated-chi-square (E-Chi) distribution. If $b=a=1$, in addition to $\tau = \alpha = 2$ and $k = \frac{p}{2}$, we obtain the well-known chi-square distribution.

2.2.5 Kum-Scaled Chi-Square distribution

If we take $\tau = 1$, $\alpha = \sqrt{2\sigma}$ and $k= \frac{p}{2}$, the density is given by

$$f(t)=\frac{2ab}{\sqrt{2}\sigma\Gamma\left(\frac{p}{2}\right)}\left(\frac{t}{\sqrt{2}\sigma}\right)^{p-1}\exp\left[-\left(\frac{t^2}{2\sigma^2}\right)\right]\left\{\gamma_1\left[\frac{p}{2},\left(\frac{t}{\sqrt{2}\sigma}\right)^2\right]\right\}^{a-1}\times\left(1-\left\{\gamma_1\left[\frac{p}{2},\left(\frac{t}{\sqrt{2}\sigma}\right)^2\right]\right\}^{a}\right)^{b-1}, \quad t>0 \tag{2.8}$$

Which is the Kum-Scaled Chi-Square (KumSChi) distribution. For $b=\tau=1$, $\alpha=\sqrt{2}\sigma$ $k = \frac{p}{2}$, we obtain the exponentiated scaled chi-square (ESChi) distribution.

If $b = a = 1$, in addition to $\alpha = \sqrt{2}\sigma, \tau = 1$ and $k = \frac{p}{2}$, the special case coincides with the scaled chi-square (SChi) distribution.

2.2.6 Kum-Maxwell distribution

If we take $\tau = 2$, $\alpha = \sqrt{\theta}$ and $k = \frac{3}{2}$, the density is given by

$$f(t) = \frac{4ab}{\sqrt{\pi}\,\theta^{3/2}} t^2 \exp\left[-\left(\frac{t^2}{\theta}\right)\right]\left\{\gamma_1\left[\frac{3}{2},\left(\frac{t^2}{\theta}\right)\right]\right\}^{a-1} \times \left(1-\left\{\gamma_1\left[\frac{3}{2},\left(\frac{t^2}{\theta}\right)\right]\right\}^{a}\right)^{b-1}, \quad t>0$$

(2.9)

Which is the Kum-Maxwell (KumMa) distribution. For $b = a, \tau = 2, \alpha = \sqrt{\theta}$, and $k = \frac{3}{2}$, we obtain the exponentiated Maxwell (EM) distribution. If $b = a = 1$ in addition to $\alpha = \sqrt{\theta}$, $\tau = 2$, $k - \frac{3}{2}$, it reduces to the Maxwell (Ma) distribution (see, for example, Bekker and Roux, 2005) [.6]

2.2.7 Kum-Nakagami distribution

If we take $\tau = 2$, $\alpha = \sqrt{\frac{w}{\mu}}$ and $k = \mu$, the density is given by

$$f(t) = \frac{2\mu^{\mu} ab}{w^{\mu}\Gamma(\mu)} t^{2\mu-1} \exp\left[-\left(\frac{\mu t^2}{w}\right)\right]\left\{\gamma_1\left[\mu,\left(\frac{\mu t^2}{w}\right)\right]\right\}^{a-1} \times \left(1-\left\{\gamma_1\left[\mu,\left(\frac{\mu t^2}{w}\right)\right]^{a}\right\}\right)^{b-1}, \quad t>0$$

(2.10)

Which is the Kum-Nakagami (KumNa) distribution. For b=1, τ=2, $\alpha = \sqrt{\frac{\omega}{\mu}}$ and k=μ, we obtain the exponentiated Nakagami (EM) distribution. If b=a=1, in addition to $\alpha = \sqrt{\frac{\omega}{\mu}}$, τ=2 and k=μ, the special case corresponds to the Nakagami (Na) distribution (see,for example, Shankar et al.2001)[65].

2.2.8 Kum-Generalized Half-Normal distribution

If τ=2γ, $\alpha = 2^{\frac{1}{2\gamma}}\theta$ and $k = \frac{1}{2}$, the KumGG distribution becomes

$$f(t) = \frac{ab\gamma}{t}\sqrt{\frac{2}{\pi}}\left(\frac{t}{\theta}\right)^{\gamma} \exp\left[-\frac{1}{2}\left(\frac{t}{\theta}\right)^{2\gamma}\right]\left\{\gamma_1\left[\frac{1}{2},\frac{1}{2}\left(\frac{t}{\theta}\right)^{2\gamma}\right]\right\}^{a-1} \times \left(1-\left\{\gamma_1\left[\frac{1}{2},\frac{1}{2}\left(\frac{t}{\theta}\right)^{2\gamma}\right]^{a}\right\}\right)^{b-1}, \quad t>0 \tag{2.11}$$

Which is referred to as the Kum-generalized half-normal (KumGHN) distribution. For b=1, τ=2γ, $\alpha = 2^{\frac{1}{2\gamma}}\theta$ and $k = \frac{1}{2}$, we obtain the exponentiated generalized half-normal (EGHN) distribution. For $\alpha = 2^{\frac{1}{2\gamma}}\theta, \tau = k = 2$, we obtain the Kum-half normal (KumHN) distribution. If b=1, $\alpha = 2^{\frac{1}{2}}\theta, \tau = 2$ and $k = \frac{1}{2}$, the reduced model is colled the exponentiated half-normal (EHN) distribution. If b=a=1, in addition to $\alpha = 2^{\frac{1}{2\gamma}}\theta, \tau = 2\gamma, k = 2,$ the reduced model becomes the generalized half-normal (GHN) distribution introduced by Cooray and Ananda (2008)[10]. Further, if b=a=1in addition to $\alpha = 2^{\frac{1}{2}}\theta, \tau = 2, k = \frac{1}{2}$, it reduces to the well-known half-normal (HN) distribution.

2.3-Moments

We hardly need to emphasize the necessity and importance of moments in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis).In this section, we give two different expansions for determining the moments of the KumGG distribution. Let

$\mu'_r = E(T^r)$ be the r^{th} ordinary moment of the KumGG distribution. First, we obtain an infinite sum representation for μ'_r from equation (2.1). The r^{th} moment of the GG (α, β, k) distribution is

$$\mu_r{}',GG = \frac{\alpha^r \Gamma(k + r/\beta)}{\Gamma(k)}$$

Equation (2.1) then immediately gives

$$(2.12)\quad \mu_r{}' = \frac{ab\alpha^r}{\Gamma(k)} \sum_{j,l=0}^{\infty} \sum_{m=0}^{l} w_{j,l,m} I(k, \frac{r}{\tau}, m)$$

Where

$$w_{j,l,m} = \frac{(-1)^{j+l+m} \Gamma b}{\Gamma(b-j) j!} \binom{a(1+j)-1}{l} \binom{l}{m}$$

And

$$I(k, \frac{r}{\tau}, m) = \int_0^{+\infty} x^{k+\frac{r}{\tau}-1} \exp(-x) \gamma_1(k, x)^m dx .$$

For b=1 and j=0, we obtain the same result as in Cordeiro et al. (2009)[13]. For calculating the last integral, using the series expansion for the incomplete gamma function yields.

$$I(k, \frac{r}{\tau}, m) = \int_0^{+\infty} x^{k+\frac{r}{\tau}-1} \exp(-x) \left[x^k \sum_{p=0}^{\infty} \frac{(-x)p}{(k+p)p!} \right] dx .$$

This integral can be determined from equations (24) and (25) of Nadarajah (2008b)[54] in terms of the Lauricella function of type A (Exton, 1978[22]; Aarset, 0987[1]) defined by

$$F_A^{(n)}(a;b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)=\sum_{m_1}^{\infty}\ldots\sum_{m_n}^{\infty}\frac{(a)m_1+\ldots+m_n(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}}\frac{x_1^{m_1}\ldots x_n^{m_n}}{m_1!\ldots m_n!},$$

(2.13)

=1) $(a)_0$ is the ascending factorial defined by (with the convention that $(a)_i$ where

$$(a)_i = a(a+1)\ldots(a+i-1)$$

Numerical routines for the direct computation of the lauricella function of type A are available, see Exton (1978)[22] and Mathematica Trott (2006)[69]. We obtain

$$I(k,\frac{r}{\tau},m)=k^{-m}\Gamma\left(r/\tau+k(m+1)\right)\times F_A^{(m)}(r/\tau+k(m+1);k;k+1,\ldots;k+1;-1,\ldots,-1).$$

(2.14)

The moments of the KumGG distribution in (2.4) are the main results of this section. Graphical representation of the skewness and kurtosis when α=0.5, τ=0.08 and k=3, as a function of a for selected values of b, and as a function of b for some choices of a, are given in figures 4 and 5, respectively.

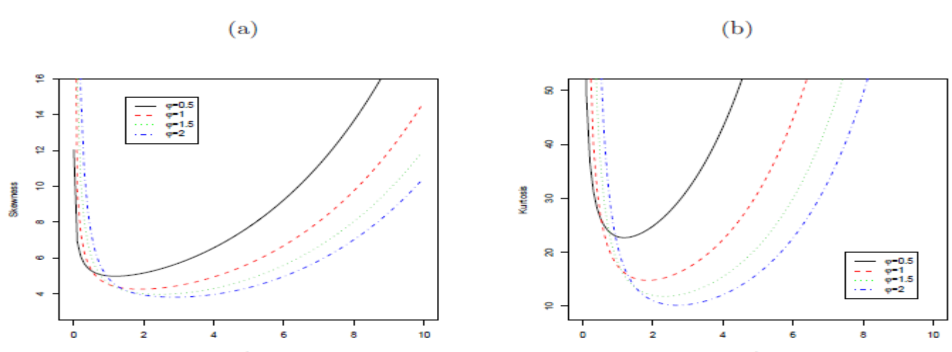


Figure 6: Skewness and kurtosis of the KumGG distribution as a function of the parameter a for selected values of b.

2.4 Moment Generating Function

Let T be a random variable having the KummGG(α,τ,k,a,b) density function (2.1). We now derive a closed form expression for the mgf, say M(s)=E[exp(sT)], of T. First, we obtain the mgf of the GG(α,τ,k) distribution. We have

$$M_{\alpha,\tau,K}(s) = \frac{\tau}{\alpha^{\tau k}\Gamma(k)}\int_0^{\infty} \exp(st)t^{\tau k-1}\exp\left\{-(t/\alpha)^{\tau}\right\}dt. \quad (2.15)$$

setting u = t , we obtain

$$M_{\alpha,\tau,K}(s) = \frac{\tau}{\Gamma(k)}\int_0^{\infty} \exp(s\alpha u)u^{\tau k-1}\exp(-u^{\tau})du.$$

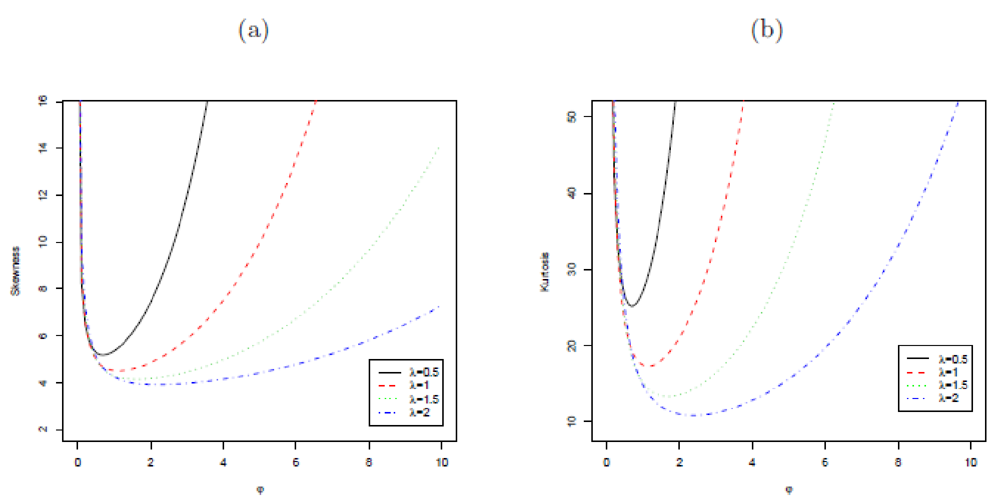
Expanding the exponential in Taylor series, we have

$$(2.16)\ M_{\alpha,\tau,K}(s) = \frac{\tau}{\Gamma(k)}\sum_{m=0}^{\infty}\frac{(\alpha s)^m}{m!}\int_0^{\infty} u^{\tau k+m-1}\exp(-u^{\tau})du.$$

where

$$\int_0^{\infty} u^{\tau k+m-1}\exp(-u^{\tau})du = \tau^{-1}\Gamma\left(\frac{k+m}{\tau}\right) \quad \text{and then}$$

$$M_{\alpha,\tau,k}(s) = \frac{1}{\Gamma(k)}\sum_{m=0}^{\infty}\Gamma\left(\frac{m}{\tau}+k\right)\frac{(\alpha s)^m}{m!}$$



Figures 5: Skewness and kurtosis of the parameter as a function of the parameter b for selected values of a.

2.5 Maximum Likelihood Estimation

Let T_i be a random variable following (2.1) with the vector of parameters

The data encountered in survival analysis and reliability studies $\theta = (\alpha, \tau, k, a, b)^T$. are often censored. A very simple random censoring mechanism that is often realistic is one in which each individual i is assumed to have a lifetime T_i and a censoring time C_i, where T_i and C_i are independent random variables. Suppose that the data consist of $= \min(T_i, C_i)$ for i=1,…,n . The censored log- t_i n independent observations for the model parameters is $l(\theta)$ likelihood

$$l(\theta) = \tau \log\left[\frac{ab\tau}{\alpha\Gamma(k)}\right] - \sum_{i\in F}\left(\frac{t_i}{\alpha}\right)^{\tau} + (\tau k - 1)\sum_{i\in F}\log\left(\frac{t_i}{\alpha}\right) + (a-1)\sum_{i\in F}\log\left\{\eta\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}$$

$$+ (b-1)\sum_{i\in F}\log\left(1 - \left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{a}\right) + b\sum_{i\in C}\log\left(1 - \left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{a}\right) \quad (2.17)$$

where r is the number of failures and F and C denote the uncensored and censored sets of observations, respectively. The score components corresponding to the parameters in θ are given by

$$U_\alpha(\theta) = -\frac{r\,\tau\,k}{\alpha} + \frac{\tau}{\alpha}\sum_{i\in F}u_i - \frac{\tau}{\alpha}\sum_{i\in F}u_i s_i + \frac{a\tau(b-1)}{\alpha\Gamma(k)}\sum_{i\in F}u_i p_i + \frac{a\tau b}{\alpha\Gamma(k)}\sum_{i\in C}u_i p_i$$

$$U_\tau(\theta) = \frac{r}{\tau} - \frac{1}{\tau}\sum_{i\in F}u_i\log(u_i) + \frac{k}{\tau}\sum_{i\in F}\log(u_i) + \frac{1}{\tau}\sum_{i\in F}\upsilon_i s_i\log(u_i)$$
$$-\frac{a(b-1)}{\tau}\sum_{i\in F}\upsilon_i p_i\log(u_i) - \frac{ab}{\tau}\sum_{i\in C}\upsilon_i p_i\log(u_i)$$

$$U_k(\theta) = -ra\Psi(k) + \sum_{i\in F}\log(u_i) + \sum_{i\in F}s_i q_i + (b-1)\left[ra\Psi(k)\right]\sum_{i\in F}p_i\gamma_1(k,u_i)$$
$$-a(b-1)\sum_{i\in F}p_i q_i + ab\Psi(k)(n-r-1)\sum_{i\in C}p_i\gamma_1(k,u_i) - ab\sum_{i\in F}p_i q_i$$

and $U_a(\theta)=\frac{r}{a}+\sum_{i\in F}\log[\gamma_1(k,u_i)]-(b-1)\sum_{i\in F}b_i[\gamma_1(k,u_i)]^a-b\sum_{i\in C}b_i[\gamma_1(k,u_i)]^a$

$$U_b(\theta)=\frac{r}{b}+\sum_{i=1}^{n}\log(\omega_i) \tag{2.18}$$

Where

$$u_i=\left(\frac{t}{\alpha}\right)^{\tau}\ ,\ g_i=u_i^k\exp(-u_i)\,,\ \omega_i=1-\eta(k,u_i)^a\ ,\upsilon_i=\frac{g_i}{\Gamma(k)}$$

$$s_i=\frac{(a-1)}{\gamma_1(k,u_i)}\quad ,\ [\gamma(k,u_i)]_k=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}J(u_i,k+n-1,1)$$

$$p_i=\frac{\gamma_1(k,u_i)^{a-1}}{\omega_i},\ q_i=\frac{[\gamma(k,u_i)]_k}{\Gamma(k)}\quad ,\ \mathrm{b_i}=\frac{\log[\gamma_1(k,u_i)]}{\omega_i}$$

$\psi(.)$ is the digamma function and $J(u_i,k+n-1,1)$ is defied in Appendix C in Marcelino A. R. Pascoa et al (2011)[50].

For interval estimation and hypothesis tests on the model parameters, we require the 5×5 unit observed information matrix

$$J=j(\theta)=\begin{pmatrix} j_{\alpha,\alpha} & j_{\alpha,\tau} & j_{\alpha,k} & j_{\alpha,a} & j_{\alpha,b}\\ j_{\tau,\alpha} & j_{\tau,\tau} & j_{\tau,k} & j_{\tau,a} & j_{\tau,b}\\ j_{k,\alpha} & j_{k,\tau} & j_{k,k} & j_{k,a} & j_{k,b}\\ j_{a,\alpha} & j_{a,\tau} & j_{a,k} & j_{a,a} & j_{a,b}\\ j_{b,\alpha} & j_{b,\tau} & j_{b,k} & j_{b,a} & j_{b,b}\end{pmatrix}$$

the MLE $\hat{\theta}$ of θ is obtained numerically from the nonlinear equations $U_\alpha(\theta)=U_\tau(\theta)=U_k(\theta)=U_a(\theta)=U_b(\theta)=0$

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\theta}-\theta)$ *is* $N_5(0,I(\theta)^{-1})$ where $I(\theta)$ is the unit expected information matrix. This approximated distribution holds when $I(\theta)$ is replaced by $J(\hat{\theta})$, i.e., the observed information matrix evaluated at $\hat{\theta}$.The multivariate normal $N_5(0,J(\hat{\theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the interval with significance level γ for each parameter θ_r is given by

$$ACI\ (\theta_r, 100\ (1-\gamma)\%) = \left(\hat{\theta}_r - z_{\gamma/2}\sqrt{\frac{\hat{j}\theta_r, \theta_r}{n}}, \hat{\theta}_r + z_{\gamma/2}\sqrt{\frac{\hat{j}\theta_r, \theta_r}{n}}\right) \tag{2.19}$$

where $\hat{j}^{\theta_r,\theta_r}$ is the estimated r^{th} diagonal element of $J(\hat{\theta})^{-1}$ for r =1,…,5 and $z_{\gamma/2}$ is the quantile 1-$\gamma/2$ of the standard normal distribution.

The likelihoods ratio (LR) statistic is useful for testing goodness-of-fit of the KumGG distribution and for comparing it with some of its special sub-models (see section 3). We can compute the maximum values of the unrestricted and restricted long-likelihoods to construct LR statistics for testing some sub-models of the Kum GG distribution. For example, we may use the LR statistic to check if the fit using the Kum GG distribution is statistically "superior" to a fit using the Kum GHN, KumSChi, GG and KumW distributions for a given data set. In any case, hypothesis tests of the type $H_0 : \theta = \theta_0$ versus $H : \theta \neq \theta_0$ where θ_0 is a specified vector, can be performed using any of the above three asymptotically equivalent statistic. For example, the test of $H_0 : b = 1$ versus $H : H_0$ not true is equivalent to compare the EGG distribution with the Kum GG distribution for which the Lr statistic reduces to $w = 2\left[\ell\left(\hat{\alpha}, \hat{\tau}, \hat{k}, \hat{a}, \hat{b}\right) - \ell(\tilde{\alpha}, \tilde{\tau}, k, \tilde{a}, 1\right]$ where $\hat{\alpha}, \hat{\tau}, \hat{k}, \hat{a}$ and $\hat{b}$ are the MLEs under H and $\hat{\alpha}, \hat{\tau}, \hat{k}$ and $\hat{a}$ are the estimates under H_0.

3-The Kumaraswamy Generalized Pareto Distribution

3.1 Introduction

The generalized Pareto (GP) distribution is the most widely applied model for univariate extreme values. Possible applications cover most areas of science engineering and medicine. Some published applications are: lifetime data analysis, coupon collector's problem, analysis of radio audience data, analysis of rainfall time series, comparing time series, comparing investment risk between Chinese and American stock markets, regional flood frequency analysis, drought modeling, value at risk, analysis of turbine steady- state second- order material property closures, wind extremes, analysis of a Spanish motor liability insurance database, analysis of finite bu erqueues, river flow modeling, measuring liquidity risk of open-end funds, modeling of extreme earthquake events, estimation of the maximum inclusion size in clean steels, and modeling of high- concent-Rations in short-range atmospheric dispersion.

For details on the GP distribution, its theory and further applicationswe refer the readers to Leadbetter et al. (1987)[42], Embrechts et al (1997)[20], Castillo et al. (2005)[9], and Resnick (2008)[62].

However, the GP distribution has been misused in too many areas, as can be seen from the list given. It does not give adequate fits inmany areas. For example, Madsen and osbjerg (1998)[48] find hat the GP distribution does not give due to a good fit to drough a good fit to drought deficit volumes many small drought events, Joshi (2010)[35] finds

"Both plots indicate that the(exponential), Pareto, and Gpd (generalized Pareto distributions) are a poor fit".

The GP distribution has been widely used to model lifetimes: see, for example, Mahmoudi (2011)[49].

The Kum-exponential distribution has been used to model lifetimes, see Cordeiro et al. (2010)[12].

There are other ways to generalize the GP distribution. The most recent generalizations of the GP distribution were proposed by

Papastathopoulos and Tawn (2013)[58]. They referred to their generalizations as EGP1, EGP2 and EGP3 distributions.

The EGP1 distribution is specified by the cumulative distribution function

$$F(x) = \frac{1}{B(\kappa, 1/\mid \xi \mid)} B_{1-\left(1+\xi \frac{x}{\sigma}\right)^{-|\xi|/\xi}} (\kappa, 1/\mid \xi \mid), \tag{3.1}$$

for $x > 0$ (if $\xi \geq 0$), $0 < x \leq -\sigma/\xi$ (if $\xi < 0$), $\sigma > 0$, $\kappa > 0$ and $-\infty < \xi < \infty$, where $B_x(\cdot,\cdot)$ denotes the incomplete beta function defined by

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt,$$

and $B(\cdot,\cdot)$ denotes the beta function defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt.$$

The EGP2 distribution is specified by the cumulative distribution function

$$F(x) = \frac{1}{\Gamma(\kappa)}\gamma\left[\kappa, \frac{1}{\xi}\ln\left(1+\xi\frac{x}{\sigma}\right)\right], \tag{3.2}$$

for $x > 0$ (if $\xi \geq 0$), $0 < x \leq -\sigma/\xi$ (if $\xi < 0$), $\sigma > 0$, $\kappa > 0$ and $-\infty < \xi < \infty$, where $B_x(\cdot,\cdot)$ denotes the incomplete - beta function defined by

$$\Gamma(a) = \int_0^\infty t^{a-1}\exp(-t)dt,$$

and $\gamma(\cdot,\cdot)$ denotes the incomplete gamma function defined by

$$\gamma(a,x) = \int_0^x t^{a-1}\exp(-t)dt.$$

The EGP3 distribution is specified by the cumulative distribution function

$$F(x) = \left\{1 - \left(1 + \xi\frac{x}{\sigma}\right)^{-1/\xi}\right\}^{\kappa}, \quad (3.3).$$

For $x > 0$ (if $\xi \geq 0$), $0 < x \leq -\sigma/\xi$ (if $\xi < 0$), $\sigma > 0$, $\kappa > 0$ and $-\infty < \xi < \infty$.

Unfortunately, none of the distributions given by (3.1) - (3.3) are new. There have been many published papers (possibly in hundreds) proposing distributions same as (3.1) - (3.3) or containing (3.1) - (3.3) as special cases. Besides, the distributions given by Papastathopoulos and Tawn (2013)[56] appear complicated: at least (3.1) and (3.2) involve the incomplete beta function and the incomplete gamma function , special functions requiring numerical routines.

now explain why the distributions given by (3.1) -(3.3) are not new. Firstly, (3.1) is a special case of the class of beta - G distributions introduced by Eugene et al. (2002)[21] and followed by Jones (2004)[32] and many others. The beta - G distribution is specified bythe cumulative distribution function

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1} dt, \quad (3.4)$$

for $a > 0$ and $b > 0$. Note that (3.1) is a special case of (3.4) for $G(\cdot)$ specified by

$$G(x) = 1 - \left(1 + \xi\frac{x}{\sigma}\right)^{-|\xi|/\xi}.$$

This special case is considered in detail by Akinsete et al.(2008, Section 2.2),Mah moudi (2011)[49] and many others. Secondly, (3.2) is a specia l case of the class of gamma- G distributions introduced by Zografos and Balakrishnan (2009)[7 4] and followed by Ristic and Balakrishnan(2012)[63], Nadarajah et al. (2012)[55] and many others. The gamma – G distribution is specified by the cumulative distribution function.

$$F(x) = \frac{\gamma\left(a, -\log\left[1 - G(x)\right]\right)}{\Gamma(a)}, \qquad (3.5)$$

for a>0. Note that (3.2) is a special case of (3.5) for G(·) a GP cumulative distribution function. Furthermore, the formula for the cumulative distribution function of the EGP2 distribution given in Papastathopoulos and Tawn (2013)[58]is not a valid cumulative distribution function! Finally, (3.3) is identical to the exponentiated Pareto distribution studied by Afify(2010)[3] and many others.

Now, we study the mathematical properties of the KumGP distribution. From now on, we write the cumulative distribution function and the probability density function of the GP distribution by

$$G_{\xi,\sigma}(x) = 1 - u \qquad (3.6)$$

and

$$g_{\xi,\sigma}(x) = \sigma^{-1} u^{1+\xi}, \qquad (3.7)$$

respectively, where u = $\{1+\xi(x-t)/\sigma\}^{-1/\xi}$. The cumulative distribu-tion function and the probability density function of the KumGP distribution can be written as

$$F(x) = 1 - \{1 - (1-u)^a\}^b, \qquad (3.8)$$

and

$$f(x) = \sigma^{-1} ab u^{1+\xi} (1-u)^{a-1} \{1 - (1-u)^a\}^{b-1}, \qquad (3.9)$$

respectively. The EGP3 distribution given by (3.3) is a particular case of the Kum - GP distribution. Unlike the EGP1 and EGP2 distributions, the Kum GP distribution does not involve special functions. So, one can expect that the KumGP distribution could attract wider applicability than the EGP1, EGP2 and EGP3 distri- butions.

The KumGP distribution given by (3.9) is much more flexible than the GP distribution and can allow for greater flexibility of tails. Plots of the probability density function in (3.9) for some parameter values are density function in (3.9) for some parameter values are given in Figure 6.

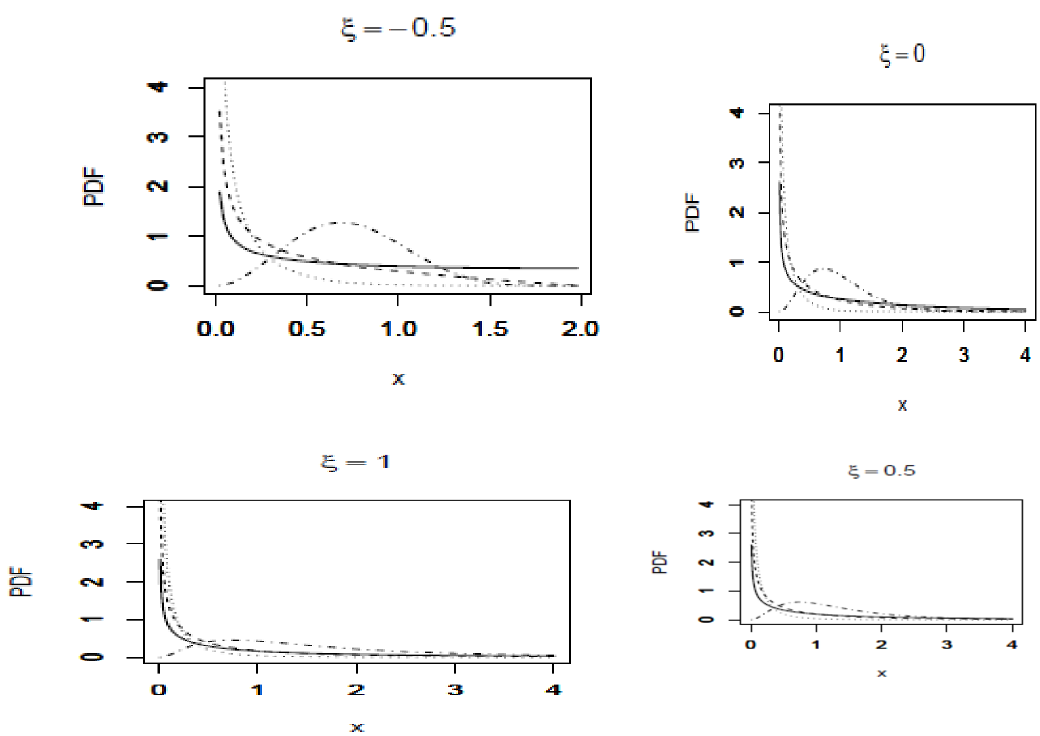


Figure 6: Plots of (15) for u = 0, σ = 1, (a, b) = (0.5, 0.5) (solidcurve), (a, b) = (0.5, 1) (curve of dashes), (a, b) = (0.5, 3) (curve of dots) and (a, b) = (3, 3) (curve of dots and dashes)

If X is a random variable with probability density function, (3.9) we write

$X \sim$ KumGP(a, b, σ, ξ). The KumGP quantile function is

obtained by inverting (3.8):

$$x = Q(z) = F^{-1}(z) = t + \frac{\sigma}{\xi}\left\{\left[1 - \left\{1 - (1-z)^{1/b}\right\}^{1/a}\right]^{-\xi} - 1\right\}. \tag{3.10}$$

(So, one can generate KumGP variates from (16) by setting $X = Q(U)$, where U is a uniform variate on the unit interval (0, 1).

3.2 Shape of Probability Density Function

The first derivative of $\log\{f(x)\}$ for the KumGP distribution is:

$$\frac{d\log f(x)}{dx} = -\frac{u^{1+\xi}}{\sigma}\left\{\frac{1+\xi}{u} - \frac{a-1}{1-u} + \frac{a(b-1)(1-u)^{a-1}}{1-(1-u)^a}\right\},$$

where $u = \{1 + \xi(x - t)/\sigma\}^{-1/\xi}$. So, the modes of $f(x)$ are the roots of the equation

$$\frac{a(b-1)(1-u)^{a-1}}{1-(1-u)^a} = \frac{a-1}{1-u} - \frac{1+\xi}{u}. \tag{3.11}$$

There may be more than one root to (3.11).

Furthermore, the asymptotes of $f(x)$ and $F(x)$ as $u \to 0, 1$ are given by

$$f(x) \sim a^b b\sigma^{-1} u^{b+\xi},$$

$$f(x) \sim ab\sigma^{-1}(1-u)^{a-1},$$

as $u \to 1$,

$$1 - F(x) \sim (au)^b,$$

$$F(x) \sim b(1 - u)^a,$$

as u →

1. Note that both the upper and lower tails of $f(x)$ are polynomials with respect to u. Larger values of a correspond to heavier upper tails of f. Larger values of b correspond to lighter upper tails of f.

Plots of the shapes of (3.9) for $t = 0$, $\sigma = 1$ and selected values of (a, b, ξ) are given in Figure 6.Both unimodal and monotonically decreasing shapes appear possible. Unimodal shapes appear when both a and b are large. Monotonically decreasing shapes appear when either a or b is small.

3.3 Shape of Hazard Rate Function

The hazard rate function defined by $h(x) = f(x)/\{1 - F(x)\}$ is an important quantity characterizing life phenomena of a system. For the KumGP distribution, $h(x)$ takes the form

$$h(x) = \frac{abu^{1+\xi}(1-u)^{a-1}}{\sigma\left[1-(1-u)^a\right]}, \tag{3.12}$$

where $u = \{1 + \xi(x - t)/\sigma\}^{-1/\xi}$. The first derivative of $\log h(x)$ is:

$$\frac{d\log h(x)}{dx} = -\frac{u^{1+\xi}}{\sigma}\left[\frac{1+\xi}{u} - \frac{a-1}{1-u} + \frac{a(1-u)^{a-1}}{1-(1-u)^a}\right].$$

So, the modes of $h(x)$ are t he roots of the equation

$$\frac{a(1-u)^{a-1}}{1-(1-u)^a} = \frac{a-1}{1-u} - \frac{1+\xi}{u}. \tag{3.13}$$

There may be more than one root to (3.13).

Furthermore, the asymptotes of $h(x)$ as $u \to 0, 1$ are given by

$$h(x) \sim b\sigma^{-1}u^{\xi},$$

$$h(x) \sim ab\sigma^{-1}(1-u)^{a-1},$$

as $u \to 1$. Note that both the upper and lower tails of h(x) are polynom-ials With respect to u. Larger values of a correspond to lighter lower respect to u. Larger values of a correspond to lighter lower tails. Larger values of b correspond to heavier lower tails and heavier.

Figure 7 illustrates some of the possible shapes of h(x) for t = 0, σ= 1 and selected values of (a, b, ξ). Both monotonically increasing, Monotonically decreasing and bathtub shapes appear possible. Bathtub shapes appear for negative values of ξ. Monotonically increasing shapes appear when both a and b are large. Monotonically decreasing shapes appear when either a or b is small and ξ is not negative.

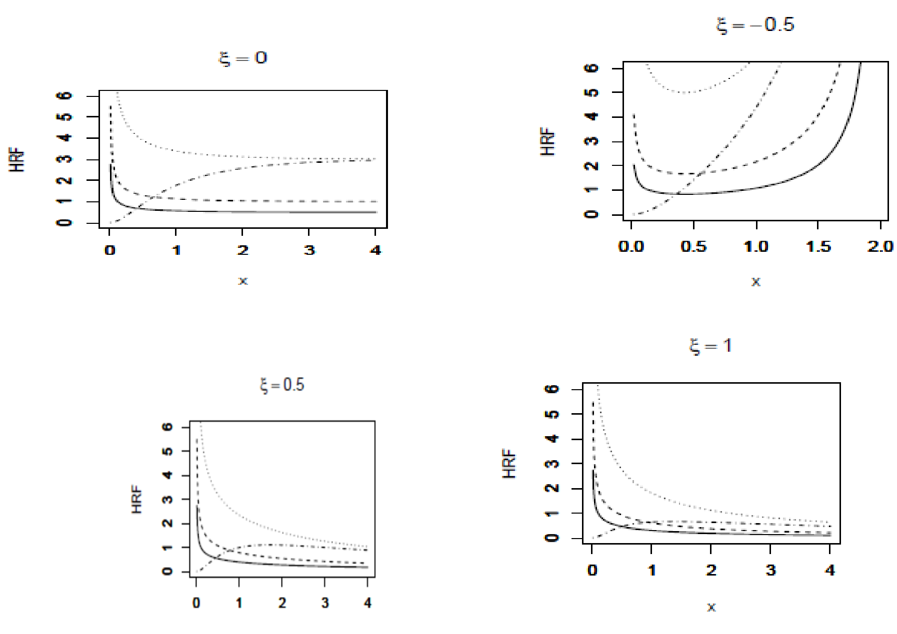


Figure 7:Plots of (18) for u = 0, σ = 1, (a, b) = (0.5, 0.5) (solid curve), (a, b) = (0.5, 1) (curve of dashes), (a, b) = (0.5, 3) (curve of dots) and (a, b) = (3, 3) (curve of dots and dashes)

Bathtub shaped hazard rates are the most realistic ones in practice. It is interesting to note that the KumGP distribution can exhibit this shape. The GP distribution cannot exhibit bathtub shaped hazard rates.

3.4 Moments

Let $X \sim \text{KumGP}(a, b, \sigma, \xi)$. Using the transformation $u = \{1+\xi(x-t)/\sigma\}^{-1/\xi}$ we can write

$$E\left(X^n\right) = ab\sum_{i=0}^{n} \binom{n}{i}\left(\frac{\sigma}{\xi}\right)^i\left(t-\frac{\sigma}{\xi}\right)^{n-i} \sum_{j=0}^{\infty}\binom{b-1}{j} \times (-1)^j B\left(1-i\xi, a+aj\right) \tag{3.14}$$

for $n \geq 1$ provided that $1 - i\xi$ is not an integer for all $i = 0, 1, \cdots, n$. The first four moments are:

$$E(X) = ab\left[\left(t-\frac{\sigma}{\xi}\right)\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j}{a+aj} + \frac{\sigma}{\xi}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^j B(1-\xi, a+aj)\right], \tag{3.15}$$

$$E(X^2) = ab\left[\left(t-\frac{\sigma}{\xi}\right)^2\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j}{a+aj} + 2\left(t-\frac{\sigma}{\xi}\right)\frac{\sigma}{\xi}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^j B(1-\xi, a+aj) + \left(\frac{\sigma}{\xi}\right)^2\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^j B(1-2\xi, a+aj)\right], \tag{3.16}$$

$$E\left(X^{3}\right)=\mathrm{ab}\Bigg[\left(t-\frac{\sigma}{\xi}\right)^{3}\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^{j}}{\mathrm{a}+\mathrm{aj}}
+3\left(t-\frac{\sigma}{\xi}\right)^{2}\frac{\sigma}{\xi}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^{j}B(1-\xi,\mathrm{a}+\mathrm{aj})
+3\left(t-\frac{\sigma}{\xi}\right)\left(\frac{\sigma}{\xi}\right)^{2}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^{j}B(1-2\xi,\mathrm{a}+\mathrm{aj})
+\left(\frac{\sigma}{\xi}\right)^{2}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^{j}B(1-3\xi,\mathrm{a}+\mathrm{aj})\Bigg] \quad (3.17)$$

and

$$E\left(X^{4}\right)=\mathrm{ab}\Bigg[\left(t-\frac{\sigma}{\xi}\right)^{4}\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^{j}}{\mathrm{a}+\mathrm{aj}}
+4\left(t-\frac{\sigma}{\xi}\right)^{3}\frac{\sigma}{\xi}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^{j}B(1-\xi,\mathrm{a}+\mathrm{aj})
+6\left(t-\frac{\sigma}{\xi}\right)^{2}\left(\frac{\sigma}{\xi}\right)^{2}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^{j}B(1-2\xi,\mathrm{a}+\mathrm{aj})
+4\left(t-\frac{\sigma}{\xi}\right)\left(\frac{\sigma}{\xi}\right)^{3}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^{j}B(1-3\xi,\mathrm{a}+\mathrm{aj})
+\left(\frac{\sigma}{\xi}\right)^{4}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^{j}B(1-4\xi,\mathrm{a}+\mathrm{aj})\Bigg], \quad (3.18)$$

provided that $1-\xi$, $1-2\xi$, $1-3\xi$ and $1-4\xi$ are not integers. The infinite series in (3.14)-(3.18) all converge.

The expressions given by (3.15) - (3.18) can be used to compute the mean, variance, skewness and kurtosis of X. The values of these four quantities versus ξ are plotted in Figure 3 for $t = 0$, $\sigma = 1$ and selected values of (a, b). It is evident each of the quantities is an increasing function of ξ for all choices of (a, b).

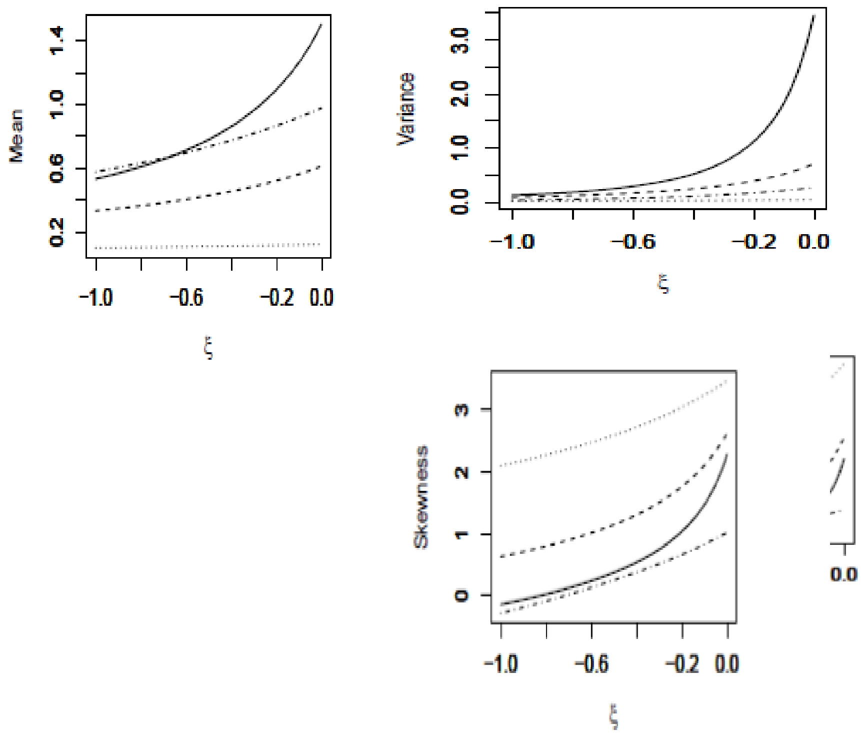


Figure 8: Mean, variance, skewness and kurtos versus ξ for $t = 0$, $\sigma = 1$, $(a,b) = (0.5, 1)$ (curve of dashes), $(a,b) = (0.5, 3)$ (curve of dots) and $(a, b) = (3, 3)$ (curve of dots and dashes)

3.5 Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the order Statistics for a random sample $X_1, X_2, \cdots, X_n$ from (3.9). Then the probability density function of the kth order statistic, say $Y = X_{k:n}$, can be expressed as

$$
\begin{aligned}
f_Y(y) &= \frac{abn!}{\sigma(k-1)!(n-k)!} u^{1+\xi}(1-u)^{a-1}\left[1-(1-u)^a\right]^{b(n-k+1)-1} \\
&\quad \times \left\{1-\left[1-(1-u)^a\right]^b\right\}^{b-1} \\
&= \frac{abn!}{\sigma(k-1)!(n-k)!} \\
&\quad \times \sum_{i=0}^{\infty} \binom{k-1}{i} (-1)^i u^{1+\xi}(1-u)^{a-1}\left[1-(1-u)^a\right]^{b(i+n-k+1)-1} \\
&= \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{\infty} \binom{k-1}{i} (-1)^i f_{a,b(i+n-k+1),\sigma,\xi}(y),
\end{aligned}
$$

where $u = \{1+\xi(y-t)/\sigma\}^{-1/\xi}$ and $f_{a,b,\sigma,\xi}(\cdot)$ denotes the probability density

function of $X_{a,b,\sigma,\xi} \sim \mathrm{KumGP}(a,b,\sigma,\xi)$. So, the probability

density function of Y is a linear combination of probability density functions of $\mathrm{KumGP}(a,b,\sigma,\xi)$. Hence, other properties of Y can be easily derived. For instance, the cumulative distribution function of Y can be expressed as

$$
F_Y(y) = \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{\infty} \binom{k-1}{i} (-1)^i F_{a,b(i+n-k+1),\sigma,\xi}(y),
$$

where $F_{a,b,\sigma,\xi}(\cdot)$ denotes the cumulative distribution function correspon-ding to $f_{a,b,\sigma,\xi}(\cdot)$. The qth moment of Y can be expressed as

$$
E\left[Y^q\right] = \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{\infty} \binom{k-1}{i} (-1)^i E\left[X^q_{a,b(i+n-k+1),\sigma,\xi}\right], \tag{3.19}
$$

where $X_{a,b,\sigma,\xi} \sim \text{KumGP}(a, b, \sigma, \xi)$.

L- moments are summary statistics for probability distributions and data samples (Hoskings, 1990[31]). They are analogous to ordinary moments but are computed from linear functions of the ordered data values. The rth L moment is defined by

$$\lambda_r = \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1}{j} \binom{r-1+j}{j} \beta_j,$$

where $\beta_j = E\{XF(X)^j\}$. In particular, $\lambda_1 = \beta_0$, $\lambda_2 = 2\beta_1 - \beta_0$, $\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0$ and $\lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0$. In general, $\beta_r = (r+1)^{-1} E(X_{r+1:r+1})$, so it can be computed using (3.19). The L moments have several advantages over ordinary moments: for example, they apply for any distribution having finite mean; no higher- order moments need be finite.

٣.6 Maximum Likelihood Estimation

Suppose $x_1, x_2, \cdots, x_n$ is a random sample of size n from (3.9). Let u_i =

$$\log L(a,b,\sigma,\xi) = n\log(ab) - n\log\sigma + (1+\xi)\sum_{i=1}^{n}\log u_i + (a-1)\sum_{i=1}^{n}\log(1-u_i) + (b-1)\sum_{i=1}^{n}\log\left[1-(1-u_i)^a\right].$$

$$\log L(a,b,\sigma,\xi) = n\log(ab) - n\log\sigma + (1+\xi)\sum_{i=1}^{n}\log u_i + (a-1)\sum_{i=1}^{n}\log(1-u_i) + (b-1)\sum_{i=1}^{n}\log\left[1-(1-u_i)^a\right].$$

(3.20)

The first- order partial derivatives of (3.20) with respect to the four parameters are:

The maximum likelihood estimates of (a, b, σ, ξ), say (a, b, σ, ξ), are the simultaneous solutions of the equations $\partial \log L / \partial a = 0$, $\partial \log L/\partial b = 0$, $\partial \log L/\partial \sigma = 0$ and $\partial \log L/\partial \xi = 0$.

As $n \to \infty$, $\sqrt{n}\,(\hat{a} - a, \hat{b} - b, \hat{\sigma} - \sigma, \hat{\xi} - \xi)$

approaches a multi variate normal vector with zero means and variance-covariance matrix, $-(\mathbf{EJ})^{-1}$, where

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial^2 \log L}{\partial a^2} & \frac{\partial^2 \log L}{\partial a \partial b} & \frac{\partial^2 \log L}{\partial a \partial \sigma} & \frac{\partial^2 \log L}{\partial a \partial \xi} \\ \frac{\partial^2 \log L}{\partial b \partial a} & \frac{\partial^2 \log L}{\partial b^2} & \frac{\partial^2 \log L}{\partial b \partial \sigma} & \frac{\partial^2 \log L}{\partial b \partial \xi} \\ \frac{\partial^2 \log L}{\partial \sigma \partial a} & \frac{\partial^2 \log L}{\partial \sigma \partial b} & \frac{\partial^2 \log L}{\partial \sigma^2} & \frac{\partial^2 \log L}{\partial \sigma \partial \xi} \\ \frac{\partial^2 \log L}{\partial \xi \partial a} & \frac{\partial^2 \log L}{\partial \xi \partial b} & \frac{\partial^2 \log L}{\partial \xi \partial \sigma} & \frac{\partial^2 \log L}{\partial \xi^2} \end{pmatrix}.$$

The matrix, $-\mathbf{EJ}$, is known as the expected information matrix.The matrix, $-\mathbf{J}$, is known as the observed information matrix. In cases of more than one maximum, we took the maximum likelihood estimates to correspond to the largest of the maxima.

In practice, n is finite. That it is best to approximate the distribution of $\sqrt{n}(\hat{a} - a, \hat{b} - b, \hat{\sigma} - \sigma, \hat{\xi} - \xi)$ by a multivariate normal distribution with zero means and variance- covariance matrix given by $-\mathbf{J}^{-1}$, inverse of the observed information matrix, with $(\hat{a}, \hat{b}, \hat{\sigma}, \hat{\xi})$ replaced (a, b, σ, ξ). So, it is useful to have explicit expressions for the elements of $\mathbf{J}$.

The multivariate normal approximation can be used to construct approximate confidence intrvals and confidence regions for the individual parameters and for the hazard and survival functions.

4-On Generalized Order Statistics from Kumaraswamy Distribution

(4.1) Introduction

The distribution of the Product and ratio of random variables find an important place in the literature and much work is done when the random variables are independent and come from a particular probability distribution.

If the random variables X_1, X_2, …, X_n are arranged in ascending order of magnitudes and then written as $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$, then $X_{(i)}$ is called the i^{th} order statistic (i = 1,2,…,n) and the order random variables are necessarily dependent. The distribution of product and quotient of the extreme order statistics and that of consecutive order statistics are useful in ranking and selection problems. Subrahmaniam(1970) [36] has made the study of product and quotient of order statistics from uniform distribution and exponential distribution , whereas Malik and Trudel(1976) [30] studied the cases when the order statistics are from Pareto, power and Weibull distribution , Recently Garg (2009) [47] has studied order Statistics from Kumaraswamy Distribution.

The subject of order statistics has been further generalized and the concept of generalized order statistics in introduced and studied by Kamps in a series of papers and books [70], [71], [72],[73]. The order Statistics, record values and sequential order statistics are special cases of generalized order Statistics. This concept is widely studied by many research workers namely Ahsanullah [43, 44, 45, 46], AbEl-Baset, Ahmed and Al-Matrofi (2006)[2], Cramer and Kamps [14, 15], Cramer, Kamps and

Ryehlik (2002, 2002, 2002 , 2004) [16, 17, 18, 19] Hosking (1990) [31] and Reiess(1989) [60].

In the present paper we shall obtain the joint distribution, distribution of product and distribution of ratio of two generalized order statistics from the family of distribution known as Kumaraswamy Distribution(1980)[38].

(4.2) Definitions

4.2.1 Generalized Order Statistics

Let F(x) denoted an absolutely continuous distribution function with density function f(x) and $X_{1,n,m,k}$, $X_{2,n,m,k}$, …, $X_{n;n,m,k}$ ($k \geq 1$, m is a real number) be n generalized order statistics. Then the joint probability density function (p. d .f) can be written as Kamps (1995) [71] $f_{1,\ldots,n}(x_1,\ldots,x_n)$

$$f_{1,\ldots,n}(x_1,\ldots,x_n) = \begin{cases} k\prod_{j=1}^{n-1}\gamma_j\prod_{i=1}^{n-1}\left[(1-F(x_i))^m f(x_i)\right](1-F(x_n))^{k-1} f(x_n), \\ \qquad\qquad \text{for } F^{-1}(0) < x_1 < \ldots < x_n < F^{-1}(1) \\ 0\,, \text{ otherwise} \end{cases}$$

Where, (4.1)

$$\gamma_j = (k + (n-j)(m+1)) \text{ and } f(x) = \frac{d\,F(x)}{dx}$$

If m = 0 and k = 1 it gives the joint p. d. f. of n ordinary order statistics $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$. If m = -1 and k = 1 it gives the joint p. d. f. of the first n upper records of the independent and identically distributed random variables. Various distributional properties of generalized order statistics are studied by Kamps (2001) [70] and that of record values by Ahsanullah (1995, 2000) [43,44], Arnold, Balakrishnan and Nagaraja (1998) [4] and Raqab (2002) [61].

of $f_{r,n,m,k}$ Further integrating out $x_1,\ldots,x_{r-1}$, $x_{r+1},\ldots, x_n$ from (4.1), we get p,d.f. [Kamps (1995) [71]] as $X_{r,n,m,k}$

$$(4.2)\quad f_{r,n,m,k}(x) = \frac{C_r}{(r-1)!}[1-F(x_r)]^{\gamma_r-1} g_m^{r-1}[F(x_r)] f(x_r)$$

Where

$$C_r = \prod_{j=1}^{r} \gamma_j \quad , \qquad \gamma_r = k+(n-r)(m+1) \qquad (4.3)$$

and

$$g_m(x) = \begin{cases} \frac{1}{m+1}\left[1-(1-x)^{m+1}\right], & m \neq -1 \\ -\ln(1-x), & m=-1, \quad x \in (0,1) \end{cases} \qquad (4.4)$$

$$\lim_{m\to -1} \frac{1}{m+1}\left[1-(1-x)^{m+1}\right] = -\ln(1-x) \quad \text{Since}$$

We shal write

and for all m $x \in (0,1)$ for all $g_m(x) = \frac{1}{m+1}\left[1-(1-x)^{m+1}\right]$

With

$$(4.5)\quad g_{-1}(x) = \lim_{m\to -1} g_m(x)$$

Also the joint distribution of of i^{th} and j^{th} generalized order statistics is given by

$$f_{i,j,n,m,k}(x) = \frac{C_j}{(i-1)!(j-i-1)!}[1-F(x_i)]^m [1-F(x_j)]^{\gamma_j-1}[g_m\ F(x_i)]^{i-1} \left[g_m(F(x_j)) - g_m(F(x))\right]^{j-i-1} f(x_i)\, f(x_j) \qquad (4.6)$$

$$0 < x_i < x_j < \infty, \quad 1 \le i < j \le n$$

ymbols

The result (4.6), on taking m=0 and k=1 reduces to the joint p. d. f. of ith and jth order statistics as given in David (1981) [28].

4.2.2 The Mellin Transform

Let (X_1,X_2) be a two dimension random variable having the joint probability density function $f(x_1,x_2)$ that is positive in the first quadrant and zero elsewhere The Mellintransform of $f(x_1,x_2)$ is defined by Fox (1957) [7] as

$$M_{s_1,s_2}[f(x_1,x_2)] = \int_0^\infty \int_0^\infty x_1^{s_1-1} x_2^{s_2-1} f(x_1,x_2)\; dx_1 dx_2$$

with the inverse (4.7)

$$f(x_1,x_2) = \int_{h-i\infty}^{h+i\infty} \int_{k-i\infty}^{k+i\infty} x_1^{-s_1} x_2^{-s_2}\; M_{s_1,s_2}[f(x_1,x_2)]\, ds_1 ds_2$$

(4.8)

under the appropriate conditions discussed by Fox. we are interested in the following two particular cases K.Subrahmaniam (1970)[36].

If $Y = X_1\ X_2$, then h(y), the p.d.f. of Y, has the Mellin transform

and if $Z = X_1 / X_2$, then h(y), the p.d.f.of Z, has the Mellin transform

$M_{s_1,s_2}[g(z)] = M_{s,-s+2}[g(z)]$

4-2-3 Fox-H-function

We shall require the following definition of Fox (1961) [8] H-function

$$H_{p,q}^{m,n}\left[x \left| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_L \theta(s)\, x^s\, ds$$

(4.9)

where $\omega = \sqrt{-1}$, $x(\neq 0)$ is a compelexvariable and $x^s = Exp[s\{\log|x| + \omega \arg x\}]$,

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)}, \tag{4.10}$$

$0 \le n \le p,\ 1 \le m \le q; \alpha_j (j = 1,\ldots,p)$ m, n, p and q are non-negative integers satisfying, are assumed to be positive quantities for standardization purpose. $\beta_j (j = 1,\ldots,q)$ and

The definition of the H-function given by (4.9) will however have meaning even if some of these quantities are zero, giving us in tern simple transformation formulas. The nature of contour L, a set of sufficient conditions for the convergence of this integral, the asymptotic expansion, some of its properties and special cases can be referred to in the book by Srivastave, Gupta and Goyal (1982) [29].

4.2.4 Joint Distribution and Distributions of Product and Ratio of Two Generalized Order Statistics

Therom1.

Generalized order statistics with (i <j)Let $X_{i;n,m,k}$ and $X_{j;n,m,k}$ be i[th] and j[th] Based on a random sample of size in from the Kumaraswamy distribution. The joint p.d.f. of these generalized order statistics is giving by:

$$f_{i,j;n,m,k}(x_i, x_j) = \begin{cases} \dfrac{C_j a^2 b^2 x_i^{a-1} x_j^{a-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2} \\ .\left(1-x_i^a\right)^{b(m+1)(l_1-l_2+j-i)-1} \left(1-x_j^a\right)^{b(\gamma_j+(m+1)l_2)-1}, \quad \text{for } m \neq -1 \\ \dfrac{k^j a^2 b^j x_i^{a-1} x_j^{a-1} \left(1-x_j^a\right)^{b(k+1)-2}}{(i-1)!(j-i-1)!\left(1-x_i^a\right)} \log\left(\dfrac{1}{1-x_i^a}\right)^{i-1} \\ .[\log\left(1-x_i^a\right) - \log\left(1-x_j^a\right)]^{j-i-1}, \quad \text{for } m = -1 \end{cases} \tag{4.11}$$

provided that a , b > 0 , $1 \le i \le j \le n$, m is a real number , k ≥ 1 , 0 are defined by (4.3). $\gamma_j \le x_i < x_j \le 1$ and C_j and

Therom2.

Let $X_{i:\,n,m,k}$ and $X_{j:\,n,m,k}$ denote the i^{th} and j^{th} generalized order statistics from a random sample of size n drown from Kumaraswamy distribution defined by (1.3). Then the probability density function of the product

$$Y = X_{i\,;\,n,\,m,\,k} X_{j\,;\,n,\,m,\,k} \qquad (4.12)$$

$$Z = \frac{X_{i;\,n,\,m,k}}{X_{j;\,n,\,m,k}} \qquad (4.13)$$

are given by

$$g_{i,j,n,m,k}(y) = \frac{C_j\, a\, b^2\, y^{a-1}\left(1-y^a\right)^{b(m+1)(j-i+\gamma_j+1)-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2}$$

$$.\left(1-y^a\right)^{b(m+1)l_1} \frac{\Gamma\left(b(m+1)(l_1-l_2+j\text{-}i)\right)\Gamma\left(b\{\gamma_j+(m+1)l_2\}\right)}{\Gamma\left(b\{(m+1)(l_1+j\text{-}i)+\gamma_j\}\right)}$$

$$._2F_1[b(\gamma_j+(m+1)l_2),\, b(m+1)(l_1-l_2+j\text{-}i)\,;\, b\{(m+1)(l_1+j\text{-}i)+\gamma_j\}\,;\, 1-y^a],$$
$$m \neq -1$$

and

$$h_{i,j;\,n,m,k}(z) = \frac{C_j\, b^2}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2} \Gamma\left(b(m+1)(l_1-l_2+j\text{-}i)\right)$$

$$.\Gamma\left(b\left((\gamma_j+(m+1)l_2\right)\right) H^{1,1}_{2,2}\left[\frac{1}{z}\left|\begin{matrix}\left(\frac{1}{a},\frac{1}{a}\right),\left(1+b(\gamma_j+(m+1)l_2)+\frac{1}{a},\frac{1}{a}\right)\\ \left(1+\frac{1}{a},\frac{1}{a}\right),\left(-b(m+1)(l_1-l_2+j\text{-}i)+\frac{1}{a},\frac{1}{a}\right)\end{matrix}\right.\right], \; m \neq -1$$

Where H[z] is the Fox H - function defined by (4.10) and j-i+

(4.14)

l_1-l_2>0, a > 0, b > 0 , 1 ≤ i< j≤ n, k≥1, m and k are real numbers and γ_j and C_j are defined by (4.3). the symbols

is $h_{i,j:n,m,k}(z)$, The p.d.f. of the ratio i.e.

$$M_{s,-s+2}[h_{i,j;n,m,k}(z)] = \frac{C_j b^2}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2}$$
$$.\beta\left(b(m+1)(l_1-l_2+j-i), \frac{s+a-1}{a} \right) \beta\left(b(\gamma_j+(m+1)l_2), \frac{-s+a+1}{a} \right) \quad (4.15)$$

4.2.5 Special Cases

1) If we take a = b = 1 in thermo 1 and 2, we get the joint p.d.f. and p.d.f. of product generalized order statistics from uniform distribution. i^{th} and j^{th} and ratio of

2) If we take j = i + 1 in thermo 1 and 2, we get the joint distribution and distribution of product and ratio of consecutive generalized order statistics based on a random sample of size n from the kumaraswamy distribution.

3) If we take i = 1, j = n in thermo 1 and 2, we get the joint distribution and distribution of product and ratio of extreme generalized order statistics based on a random sample of size n from the kumaraswamy distribution.

4) If we take n to be add say 2p +1 then putting i = p +1 and j = 2p +1 in thermo 2, we get the p.d.f. of the product and ratio of peak to median of a random sample of size 2 p + 1 of generalized order statistics as Remark. If we take m=0 and k=1 in theorems 1 and 2, then generalized order statistics reduces into order ststistics and we get the joint distribution and distribution of product and ratio of order statistics $X_{i,n}$ and $X_{n,n}$ from a sample of size n from Kumaraswamy distribution as obtained resently by the Garg (2009) [47].

References

1- Aarset, M. V. (1987), How to identify bathtub hazard rate. IEEE Transactions on Reliability, 36, 106-108.

2- Abd. El. Baset, A. Ahmad and, Bakheet N. Al. Matrofi, (2006), Reccurence relation for moments and conditional moments of generalized order statistics, Univ. of Sharjah J. of Pure and Appl. Sci., 3(3), 61-81.

3- Afify, W. M, (2010), on estimation of the exponentiated Pareto distribution under different sample schemes, Statistical Methodology 7, 77-83.

4- B. C. Arnold, N. Balakrishnan and H. N. Nagaraja, (1998),"Records", John Wiley, New York.

5- Bee, M., Benedetti, R. and Espa, G. (2009), A note on maximum likelihood estimation of a Pareto mixture, No 903, Department of Economics Working Papers from Department of Economics, University of Trento, Italia.

6- Bekker, A. and Roux, J.J. (2005), Reliability characteristics of the Maxwell distribution: a Bayes estimation study. Communications in Statistics, Theory and Methods, 34, 2169-2178.

7- C. Fox. (1957), some applications of mellin transforms to the theory of bivariate statistical distributions, Proc. Cambridge Philos. Soc., 53, 620-628.

8- C. Fox. (1961), The G-and H- function as symmetrical Fourier kernels, Trans. Amer. Math. Soc. 98, 395-429.

9- Castillo, E., Hadi, A. S., Balakrishnan, N. and Sarabia, J. M. (2005), Extreme Value and Related Models with Applications in Engineering and Science. Wiley, Hoboken, New Jersey.

10- Cooray.K. and Ananda, M. M. A. (2008), a generalization of the half-normal distribution with applications to lifetime data.Communications in Statistics, Theory and Methods.37. 13231337.

11- Cordeiro G. M. and de Castro, M. (2011), A new family of generalized distributions, Journal of Statistical Computation and Simulation, 81, 883–898.

12- Cordeiro, G. M., Ortega, E. M. M. and Nadarajah, S. (2010), The KumaraswamyWeibull distribution with application to failure data, .Journal of the Franklin Institute 347, 1399-1429.

13- Cordeiro, G. M., Ortega, E. M. M. and Silva, G. O. (2009), the exponentiated generalized gamma distribution with application to lifetime data, Journal of Statistical Computation and Simulation.In Press.

14- E. Cramer and U. Kamps, (2000), Relations for expectations of functions of generalized order statistics, J. Statist. Plann. And Inference, 89(l-2), 79-89.

15- E. Cramer and U. Kamps, (2003)," Marginal distributions of sequential and generalized order statistics, Metrika, 53(3), 289-310.

16- E. Cramer, U. Kamps and, T. Rychlik, (2002), Evaluations of expected generalised order statistics in various scale units, Applicationes Mathematical, 29(3), 285- 295.

17- E. Cramer, U. Kamps and, T. Rychlik, (2002), on the existence of moments of generalized order statistics, Statist, and aprohahility Letters, 59, 397-404.

18- E. Cramer, U. Kamps and, T. Rychlik, (2002), Moments of generalized order statistics, In: H. Langseth and B. Lindqvist (Hrsg.), Third International Conference on Mathematical Methods in Reliability, Methodology and Practice, S, 165- 168. Trondheim: NTNU.

19- E. Cramer, U. Kamps and, T. Rychlik, (2004), Unimodality of uniform generalised order statistics with application to mean bounds, Ann. Inst. Statist. Math., 56(1), 183-192.

20- Embrechts, P., Kliippelberg, C. and Mikosch, T. (1997), Modelling Extremal Events: For Insurance and Finance. Springer, Berlin.

21- Eugene, N., Lee, C. and Famoye, F. (2002), Beta-normal distribution and its applications, Communications in Statistics: Theory andMethods 31, 497–512.

22- Exton, H. (1978), Handbook of Hypergeometric Integrals: Theory, Applications, Tables, Computer Programs.New York, Halsted Press.

23- Fletcher, S.C., Ponnambalam, K. (1996), Estimation of reservoir yield and storage distribution using moment's analysis, Journal of Hydrology 182, 259–275.

24- Ganji, A., Ponnambalam, K., Khalili, D., et al. (2006), Grain yield reliability analysis with crop water demand uncertainty. Stochastic Environmental Research and Risk Assessment 20, 259–277.

25- Gauss M. Cordeiro · Saralees Nadarajah, Edwin M. M. Ortega (2012), The Kumaraswamy Gumbel distribution, Stat Methods Appl (2012) 21:139–168, DOI 10.1007/s10260-011-0183-y

26- Gupta, R. D. and Kundu, D. (2001), Exponentiated exponential distribution: an alternative to gamma and Weibull distributions, Biometrical Journal, 43, 117-130.

27- Gupta. R. D. and Kundu, D. (1999), generalized exponential distributions.Australian and New Zealand Journal of Statistics, 41, 173-188.

28- H. A. David. (1981), Order Statistics, John Wiley, New York.

29- H. M. Srivastava, K. C. Gupta and S. P. Goyal. (1982), the H-Function of One and Two Variables with Applications, South Asian Publishers, New Delhi and Madras.

30- H. Malik and R. Trudel, (1976), Distributions of product and quotient of order statistics, University of Guelph statistical Series, 1975-30. 1-25.

31- Hosking, J. R. M. (1990), L-moments: analysis and estimation of distributions using linear combinations of order statistics, Journal of the Royal Statistical Society, Series B 52, 105-124.

32- Jones, M. C. (2004), Families of distributions arising from distributions of order statistics (with discussion), Test 13, 1-43.

33- Jones, M. C. (2009), Kumaraswamy's distribution: A beta-type distribution with some tractability advantages. Statistical Methodology, 6, 70-81.

34- Joshi, M. V. (2010), modeling the severity of random events with the SAS/ETS SEVERITY procedure, Paper 325-2010. SAS Institute Inc, Cary, North Carolina.

35- K. Subrahmaniam, (1970), on some applications of Mellin transforms to statistics: Dependent random variables", SIAMJ. Appl. Math., 19(4), 658-662.

36- Koutsoyiannis, D., Xanthopoulos, T. (1989), on the parametric approach to unit hydrograph identification. Water Resources Management 3, 107–128. Ponnambalam, K., Seifi, A., Vlach, J., 2001.

37- Kumaraswamy, P. (1980), a generalized probability density function for double-bounded random processes, Journal of Hydrology, 40, 79-88.

38- Kundu, D. and Raqab, M. Z. (2005), Generalized Rayleigh distribution: different methods of estimation.Computational Statistics and Data Analysis, 49, 187-200.

39- Lawless, J. F. (1980), Inference in the generalized gamma and log-gamma distributions, Techno metrics, 22.409-419.

40- Lawless, J. P. (2003), Statistical Models and Methods for Lifetime Data, New York. John Wiley.

41- Leadbetter, M. R., Lindgren, G. and Rootzen, H. (1987), Extremes and Related, Properties of Random Sequences and Processes, Springer, New York.

42- M. Ahsanullah. (1995), Record Statistics, Nova Science Publishers, Inc. Commack, New York.

43- M. Ahsanullah, (2000), Generalised order statistics from exponential distribution, Journal of Statistical Planning and Inference, 85. 85-91.

44- M. Ahsanullah. (2004), Record Values Theory and Application, University Press of America, London and Maryland.

45- M. Ahsanullah, (2006), the generalised order statistics from exponential distribution, Pak. J. Statist., 22(2), 121-128.

46- M. Garg. (2009), on generalized order statistics from Kumaraswami distribution, Journal of Mathematical Sciences25, 153-166.

47- Madsen, H. and Rosbjerg, D. (1998), A regional Bayesian method for estimation of extreme streamflow droughts, In Statistical and Bayesian Methods in Hydrological Sciences (Edited by E. Parent, B. Bobee, P. Hubert and J. Miquel), 327-340. Studies and Reports in Hydrology, UNESCO, Paris.

48- Mahmoudi, E. (2011), the beta generalized Pareto distribution with application to lifetime data, Mathematics and Computers in Simulation 81, 2414-2430.

49- Marcelino A.R. de Pascoaa, Edwin M.M. Ortegaa, , Gauss M. Cordeiro (2011), The Kumaraswamy generalized gamma distribution with application in survival analysis, Statistical Methodology, Volume 8, Issue 5, September 2011, Pages 411–433

50- Meeker, W. Q., Escobar, L.A. (1998), Statitical Methods for Reliability Data, John Wiley, New York.

51- Mudholkar. G. S., Srivastava, D. K. and Premier, M. (1995), the exponentiatedWeibull family: A reanalysis of the bus-motor-failure data, Technometrics.37. 436-445.

52- Mudholkar. G. S., Srivastava. D. K. and Kollia, G. D. (1996), A generalization of the Weibull distribution with application to the analysis of survival data, Journal of American Statistical Association.91, 1575-1583.

53- Nadarajah, S. (2008b), Explicit expressions for moments of order statistics, Statistics and Prob ability Letters, 78, 19G-205.

54- Nadarajah, S., Cordeiro, G. M. and Ortega, E. M. M. (2012), The Zografos-Balakrishnan-G family of distributions: mathematical properties and applications, to appear in Communications in Statistics - Theory and Methods.

55- Nadarajah, S., Cordeiro, G.M. and Ortega, E.M.M. (2011), General results for the Kumaraswamy-G distribution. Journal of Statistical Computation and Simulation, DOI: 10.1080/00949655.2011.562504.

56- Nadarajah. S. and Eljabri. S (2013), Tha Kumaraswamy Generalized Pareto Distribution, Journal of Data Science, 11, 739-766.

57- Papastathopoulos, I. and Tawn, J. A. (2013), Extended generalised Pareto models for tail estimation, Journal of Statistical Planning and Inference 143, 131-143.

58- Ponnambalam.K, Seifi.A, Vlach.J. (2001), Probabilistic design of systems with general distributions of parameters, International Journal of Circuit Theory and Applications 29, 527–536.

59- R. D. Reiess. (1989), Approximate Distributions of Order Statistics, Springer, New York.

60- Raqab, M. Z. (2002), Inferences for generalised exponential distribution based on record statistics, J. Statist. Plann. And Inference, 104(2) 339-350.

61- Resnick, S. I. (2008), Extreme Values, Regular Variation, and Point Processes, Springer, New York.

62- Ristic, M. M. and Balakrishnan, N. (2012), the gamma exponentiated exponential distribution, Jornal of Statistical Computation and Simulation 82, 1191-1206.

63- Seifi, A., Ponnambalam, K., Vlach, J. (2000), Maximization of manufacturing yield of systems with arbitrary distributions of component values, Annals of Operations Research 99, 373– 383.

64- Shankar, P. M., Dumane, V. A., Reid, J. M., Genis, V., Forsberg, F., Piccoli, C. W. and Goldberg, B. B. (2001), Classification of ultrasonic b-mode images of breast masses using Nakagami distribution. IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, 48, 569-580.

65- Stacy, E. W. (1962), a generalization of the gamma distribution, Annals of Mathematical Statistics, 33, 1187-1192.

66- Stacy, E. W. and Mihrarn, G. A. (1965), Parameter estimation for a generalized gamma distribution, Technometris, 7, 349-358.

67- Sundar, V., Subbiah, K. (1989), Application of double bounded probability density-function for analysis of ocean waves, Ocean Engineering 16, 193 200.

68- Trott, M. (2006), the Mathematica Guidebook for Symbolics, With 1 DVD-ROM (Windows Macintosh and UNIX), New York, Springer.

69- U. Kamps, and E. Cramer. (2001), on the distributions of generalised order statistics, Statist.35, 269-280.

70- U. Kamps. (1995), A Concept of Generalized Order Statistics, B. G. Teubner, Stuttgart, Germany.
71- U. Kamps. (1995), A Concept of generalised order statistics, J. Statist. Plann. Inference, 48, 1-23.
72- U. Kamps. (1998), Subranges of generalised order statistics from exponential distributions, Fasciculi Mathematici, 28, 63-70.
73- Zografos, K. and Balakrishnan, N. (2009), on families of beta- and generalized gamma-generated distributions and associated inference, Statistical Methodology 6, 344-362.

ملخص: توزيع كوماراسوامي المعمم

يقدم هذا البحث المرجعي الدراسات السابقة لتوزيع كوماراسوامي المعمم ، وباستخدام أحد التوزيعات المعروفة كتوزيع جاما وتوزيع باريتو في صيغة توزيع كوماراسوامي المعمم يمكن الحصول على توزيعات معممة جديدة لها خصائص جديدة كدالة معدل الفشل والتي تعتمد على قيم المعلمات للتوزيع المعمم الجديد.ويتضمن توزيع كوماراسوامي جاما المعمم حالات خاصة لتوزيعات معممة أخرى، وكذلك أيضا توزيع كوماراسوامي باريتو المعمم يتضمن حالات خاصة لتوزيعات معممة أخرى.وتتضمن الدراسة أيضا الإحصاءات الترتيبية لتوزيع كوماراسوامي المعمم ودراسة خصائصه باستخدام طرق التقدير.