Classical and Bayesian Inference of GPW Distribution Based on Record Values: Estimation and Prediction

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ABSTRACT

In this paper, the classical and Bayesian estimation of the parameters of generalized power Weibull distribution based on record values are discussed. Lindley approximation is used to obtain explicit forms of Bayes estimators. Also, the classical and Bayesian predictions of the future record values from generalized power Weibull distribution are discussed. A real dataset and the simulation study are presented for illustration purposes.

Keywords: generalized power Weibull distribution; record values; squared error loss function; prediction; Bayes estimation; Lindley approximation; maximum likelihood estimation; application

1. INTRODUCTION

The traditional Weibull distribution is a commonly used for modeling data in reliability, and life testing studies. Because its hazard rate function is simple and can take the skewed form either, positively or negatively. However, in case the hazard rates are non-monotone (bathtub or unimodal) shapes, the Weibull distribution will be not suitable. Therefore, the researchers developed many generalizations of Weibull distribution to increase its flexibility. One of the most important of them is called the generalized power Weibull (gpw) distribution that has been introduced by (Nikulin & Haghighi, 2006).

The cumulative distribution function (cdf) and probability density function (pdf) of gpw distribution respectively, are

$$F(x;\alpha,\theta) = 1 - e^{1-(1+x^{\alpha})^{\theta}}, \quad x > 0, \ \alpha,\theta > 0 \tag{1.1}$$

and

$$f(x;\alpha,\theta) = \alpha\theta x^{\alpha-1}(1+x^{\alpha})^{\theta-1}e^{1-(1+x^{\alpha})^{\theta}}, \quad x > 0, \ \alpha,\theta > 0 \tag{1.2}$$

where $\alpha > 0$ and $\theta > 0$ are two shape parameters. The corresponding survival and failure rate functions, respectively, are

$$S(x;\alpha,\theta) = e^{1-(1+x^{\alpha})^{\theta}}, \quad x > 0, \ \alpha,\theta > 0. \tag{1.3}$$

and

$$h(x;\alpha,\theta) = \alpha\theta x^{\alpha-1}(1+x^{\alpha})^{\theta-1} \tag{1.4}$$

The quantile function of the generalized power Weibull distribution is

$$q(u;\alpha,\theta) = \left((1-\ln(1-u))^{\frac{1}{\theta}} - 1\right)^{\frac{1}{\alpha}}, \quad 0 \le u \le 1. \tag{1.5}$$

In sequential events, the event value that exceeding all previous values is of special interest and is named record value. The record values arise in all applied and scientific fields, such as sports, climate, geophysics, volcanology, hydrology, and life-test experiments. For example, during athletics, attention is normally paid to documenting results that surpass their predecessor, and because hydrologists typically track the higher flood values and meteorologists often normally deal with record temperatures that are high and low. Records are very important in some cases, including when we only want to study the value of the events that exceed the previous ones, or when observations are destroyed by experimental tests or it is impossible to obtain a complete sample. The first definition of record values and their functions is due to (Chandler, 1952). For more information about the concept and application of record values, refer, for instance, to (Arnold et al., 2011), (Nevzorov, 2001) and (Ahsanullah, 2004). In recent years, many scholars have interested in the issue of doing inferences for distributions dependent on the record values. For example, the inference based on record values for Chen distribution in (Selim, 2012), for the extreme value distribution in (Seo & Kim, 2017), for Nadarajah and Haghighi distribution in (Selim, 2018), for Lindley distribution in (Asgharzadeh et al., 2018) and for the power-exponential hazard rate distribution in (Tarvirdizade & Nematollahi, 2020).

This paper aims to consider classical and Bayesian methods for estimating the unknown parameters of the gpw distribution based on the record values data. Aims also to study the Bayesian and non-Bayesian prediction of the future record values of the gpw distribution. The rest of the paper is organized as follows, the Bayesian and non-Bayesian estimators are studied in Section 2. Bayesian and non-Bayesian predictions are discussed in Section 3. The estimation and prediction procedures are applied for the real data set in Section 4 and for simulation data in Section 5. Finally, the conclusion appears in Section 6.

2. Parameter Estimation

This section intends to find the ML and Bayesian estimators of the unknown parameters α and θ for the gpw model based on record values.

2.1. Classical estimation

In this section, the maximum likelihood approach is used to estimate the two unknown parameters of gpw distribution. Suppose $\{X_{U(i)} = x_i; 1 \leq i \leq m\}$ are the first m observed upper record values, then the likelihood function of upper records (see,) is given by

$$L(\theta \mid \underline{x}) = f(x_m)\prod_{i=1}^{m-1}\frac{f(x_i)}{1-F(x_i)} \tag{2.1}$$

where $\underline{x} = (x_1, x_2, \dots, x_m)$.

Then, the likelihood function of the m observed upper record values of $gpw(\alpha, \theta)$ distribution is

$$L(\alpha, \theta \mid \underline{x}) = (\alpha\theta)^m e^{1-(1+x_m^{\alpha})^{\theta}} \prod_{i=1}^{m} x_i^{\alpha-1}(1+x_i^{\alpha})^{\theta-1} \tag{2.2}$$

and the log of likelihood function (2.2) is

$$l(\alpha,\theta \mid \underline{x}) = m\, ln(\alpha) + m\, ln(\theta) + 1 - (1+x_m^{\alpha})^{\theta} + (\alpha-1)\sum_{i=1}^{m} ln(x_i) + (\theta-1)\sum_{i=1}^{m} ln(1+x_i^{\alpha}) \qquad (2.3)$$

Therefore, the likelihood equations are

$$\frac{\partial l}{\partial \alpha} = \frac{m}{\alpha} - \theta x_m^{\alpha}\, ln(x_m)(1+x_m^{\alpha})^{\theta-1} + \sum_{i=1}^{m} ln(x_i) + (\theta-1)\sum_{i=1}^{m}\frac{x_i^{\alpha}\, ln(x_i)}{1+x_i^{\alpha}} = 0\,, \qquad (2.4)$$

$$\frac{\partial l}{\partial \theta} = \frac{m}{\theta} - ln(1+x_m^{\alpha})(1+x_m^{\alpha})^{\theta} + \sum_{i=1}^{m} ln(1+x_i^{\alpha}) = 0 \qquad (2.5)$$

These equations can not be gotten in closed forms. Therefore, the numerical technique may be needed to find the maximum likelihood estimate of α, θ. The approximation of the confidence interval of α, θ can be obtained from the asymptotic normality of MLEs, where the asymptotic variance-covariance matrix is

$$\hat{I}(\alpha,\theta) = \begin{bmatrix} -L_{\alpha\alpha} & -L_{\alpha\theta} \\ -L_{\alpha\theta} & -L_{\theta\theta} \end{bmatrix}^{-1}_{\hat{\alpha},\hat{\theta}} = \begin{bmatrix} \hat{\sigma}_{\alpha} & \hat{\sigma}_{\alpha\theta} \\ \hat{\sigma}_{\alpha\theta} & \hat{\sigma}_{\theta} \end{bmatrix} \qquad (2.6)$$

where

$$L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2} = -\frac{m}{\alpha^2} - \theta(\theta-1)\, ln(x_m)^2 x_m^{2\alpha}(1+x_m^{\alpha})^{\theta-2} - \theta\, ln(x_m)^2 x_m^{\alpha}(1+x_m^{\alpha})^{\theta-1} + (\theta-1)\sum_{i=1}^{m}\frac{ln(x_i)^2 x_i^{\alpha}}{(1+x_i^{\alpha})^2} \qquad (2.7)$$

$$L_{\theta\theta} = \frac{\partial^2 L}{\partial \theta^2} = -\frac{m}{\theta^2} - ln(1+x_m^{\alpha})^2\,(1+x_m^{\alpha})^{\theta} \qquad (2.8)$$

$$L_{\alpha\theta} = \frac{\partial^2 L}{\partial \alpha \partial \theta} = -\theta x_m^{\alpha}\, ln(x_m)\, ln(1+x_m^{\alpha})(1+x_m^{\alpha})^{\theta-1} - \frac{x_m^{\alpha}\, ln(x_m)(1+x_m^{\alpha})^{\theta}}{1+x_m^{\alpha}} + \sum_{i=1}^{m}\frac{x_i^{\alpha}\, ln(x_i)}{1+x_i^{\alpha}} \qquad (2.9)$$

based on the asymptotic normality of maximum likelihood estimators, we can get approximate 100(1−τ)% confidence intervals for the unknown parameters, as follow

$$\hat{\alpha} \pm z_{\tau/2}\sqrt{\hat{\sigma}_{\hat{\alpha}}^{2}} \quad and \quad \hat{\theta} \pm z_{\tau/2}\sqrt{\hat{\sigma}_{\hat{\theta}}^{2}} \qquad (2.10)$$

In which, $z_{\tau/2}$ is an upper $\tau/2$ % of the standard normal distribution.

2.2. Bayesian estimation

In this section, the Bayesian approach is used to estimate the two unknown parameters α, θ. Assume that α and θ are independent and have gamma prior distributions are

$$\pi(\alpha|a,b) \propto \alpha^{a-1} e^{-b\alpha}, \qquad \alpha > 0$$

$$\pi(\theta|c,d) \propto \theta^{c-1} e^{-d\theta}, \qquad \theta > 0$$

Thus, the joint prior distribution for α and θ is

$$\pi(\alpha,\theta) \propto \alpha^{a-1}\theta^{c-1}e^{-b\alpha-d\theta}, \qquad \alpha,\theta > 0 \tag{2.11}$$

where $a,b,c,d > 0$ are hyper-parameters. By substituting Eq. (2.11) and Eq. (2.2), in Eq. (2.12) we immediately obtain the joint posterior function of α and θ as follows

$$\pi(\alpha,\theta|\underline{x}) = \frac{L(\alpha,\theta\mid\underline{x})\pi(\alpha,\theta)}{\int_0^\infty\int_0^\infty L(\alpha,\theta\mid\underline{x})\pi(\alpha,\theta)\,d\alpha d\theta} \tag{2.12}$$

$$\pi(\alpha,\theta|\underline{x}) = \frac{1}{\mathcal{R}}\alpha^{a+m-1}\theta^{c+m-1}e^{-b\alpha-d\theta-(1+x_m^{\alpha})^{\theta}}\prod_{i=1}^{m} x_i^{\alpha-1}(1+x_i^{\alpha})^{\theta-1} \tag{2.13}$$

where

$$\mathcal{R} = \int_0^\infty\int_0^\infty \alpha^{a+m-1}\theta^{c+m-1}e^{-b\alpha-d\theta-(1+x_m^{\alpha})^{\theta}}\prod_{i=1}^{m} x_i^{\alpha-1}(1+x_i^{\alpha})^{\theta-1}\,d\alpha d\theta \tag{2.14}$$

The Bayes estimator under the squared error loss function (SELF) $L(\hat{\beta};\beta) = (\hat{\beta}-\beta)^2$ is the posterior mean. Thus, the Bayes estimators of α and θ under SELF, respectively, are

$$\hat{\tilde{\alpha}} = E(\alpha|\underline{x}) = \frac{1}{\mathcal{R}}\int_0^\infty\int_0^\infty \alpha^{a+m}\theta^{c+m-1}e^{-b\alpha-d\theta-(1+x_m^{\alpha})^{\theta}}\prod_{i=1}^{m} x_i^{\alpha-1}(1+x_i^{\alpha})^{\theta-1}\,d\theta d\alpha \tag{2.15}$$

and

$$\hat{\tilde{\theta}} = E(\theta|\underline{x}) = \frac{1}{\mathcal{R}}\int_0^\infty\int_0^\infty \alpha^{a+m-1}\theta^{c+m}e^{-b\alpha-d\theta-(1+x_m^{\alpha})^{\theta}}\prod_{i=1}^{m} x_i^{\alpha-1}(1+x_i^{\alpha})^{\theta-1}\,d\alpha d\theta \tag{2.16}$$

The above ratios cannot be reduced to simple closed forms. Thence, to obtain the Bayes estimators of α and θ in closed forms we can use one of the Bayes approximation methods such as Lindley's approximation. (Lindley, 1980) introduced asymptotic expansions for the ratios of integrals. This approximation has been widely used to approximate the ratios of integrals that occurs in Bayesian analysis, see for example (Selim, 2018), (Abbas et al., 2019) and (Agiwal, 2021). Let we have a ratio of integrals as follow

$$E(u(\alpha,\theta)|x) = \frac{\int u(\alpha,\theta)\, e^{\mathcal{L}(\alpha,\theta)+\rho(\alpha,\theta)}\, du(\alpha,\theta)}{\int e^{\mathcal{L}(\alpha,\theta)+\rho(\alpha,\theta)}\, du(\alpha,\theta)} \tag{2.17}$$

where $u(\alpha,\theta)$ is any function of α and θ, and $\mathcal{L}(\alpha,\theta)$ is the logarithm of likelihood function, and $\rho(\alpha,\theta)$ is logarithm of joint prior distribution $\pi(\alpha,\theta)$. Then the ratio of integrals in (2.17) can be approximated by using Lindley's method as follows

$$\begin{aligned} E(u(\alpha,\theta)|x) = \Big[& u(\alpha,\theta) \\ & + \frac{1}{2}\big[(u_{\alpha\alpha} + 2u_\alpha \rho_{\alpha})\sigma_{\alpha\alpha} + (u_{\alpha\theta} + 2u_\alpha \rho_{\theta})\sigma_{\alpha\theta} + (u_{\theta\alpha} + 2u_\theta \rho_{\alpha})\sigma_{\theta\alpha} + (u_{\theta\theta} \\ & + 2u_\theta \rho_{\theta})\sigma_{\theta\theta}\big] \\ & + \frac{1}{2}\big[(u_\alpha \sigma_{\alpha\alpha} + u_\theta \sigma_{\alpha\theta})(\mathcal{L}_{\alpha\alpha\alpha}\sigma_{\alpha\alpha} + \mathcal{L}_{\alpha\theta\alpha}\sigma_{\alpha\theta} + \mathcal{L}_{\theta\alpha\alpha}\sigma_{\theta\alpha} + \mathcal{L}_{\theta\theta\alpha}\sigma_{\theta\theta})\big] + (u_\alpha \sigma_{\theta\alpha} \\ & + u_\theta \sigma_{\theta\theta})(\mathcal{L}_{\theta\alpha\alpha}\sigma_{\alpha\alpha} + \mathcal{L}_{\alpha\theta\theta}\sigma_{\alpha\theta} + \mathcal{L}_{\theta\alpha\theta}\sigma_{\theta\alpha} + \mathcal{L}_{\theta\theta\theta}\sigma_{\theta\theta})\Big]_{\hat{\alpha},\hat{\theta}} \end{aligned} \tag{2.18}$$

where $u_{ij} = \frac{\partial^2 u}{\partial\theta_i \partial\theta_j}$, $u_i = \frac{\partial u}{\partial\theta_i}$, $\mathcal{L}_{ijk} = \frac{\partial^3 \mathcal{L}}{\partial\theta_i \partial\theta_j \partial\theta_k}$, $\rho_j = \frac{\partial \rho}{\partial\theta_j}$, $\sigma_{ij} = \frac{-1}{\mathcal{L}_{ij}}$

Under assumption that α and θ are independent, Lindley's approximation can be reduced to a formula

$$\begin{aligned} E(u(\alpha,\theta)|x) = \Big[& u(\alpha,\theta) + \frac{1}{2}\big[(u_{\alpha\alpha} + 2u_\alpha \rho_{\alpha})\sigma_{\alpha\alpha} + (u_{\theta\theta} + 2u_\theta \rho_{\theta})\sigma_{\theta\theta}\big] \\ & + \frac{1}{2}\big[(u_\alpha \sigma_{\alpha\alpha})(\mathcal{L}_{\alpha\alpha\alpha}\sigma_{\alpha\alpha} + \mathcal{L}_{\theta\theta\alpha}\sigma_{\theta\theta})\big] + (u_\theta \sigma_{\theta\theta})(\mathcal{L}_{\theta\alpha\alpha}\sigma_{\alpha\alpha} + \mathcal{L}_{\theta\theta\theta}\sigma_{\theta\theta})\Big]_{\hat{\alpha},\hat{\theta}} \end{aligned} \tag{2.19}$$

Assuming that, $u(\alpha,\theta) = \alpha$, then $u_\alpha = 1 \; and \; u_{\alpha\alpha} = u_{\theta\theta} = u_\theta = 0.$ Therefore, the approximate Bayesian estimator of α is

$$\hat{\alpha}_{LB} = \left[\alpha + \rho_\alpha \sigma_{\alpha\alpha} + \frac{1}{2}\sigma_{\alpha\alpha}(\mathcal{L}_{\alpha\alpha\alpha}\sigma_{\alpha\alpha} + \mathcal{L}_{\theta\theta\alpha}\sigma_{\theta\theta})\right]_{\hat{\alpha},\hat{\theta}} \tag{2.20}$$

Also, in case $u(\alpha,\theta) = \theta$ then $u_\theta = 1 \; and \; u_{\alpha\alpha} = u_{\theta\theta} = u_\alpha = 0$. Therefore, the approximate Bayesian estimator of θ is

$$\hat{\theta}_{LB} = \left[\theta + \rho_\theta \sigma_{\theta\theta} + \frac{1}{2}\sigma_{\theta\theta}\left(\mathcal{L}_{\theta\theta\theta}\sigma_{\theta\theta} + \mathcal{L}_{\alpha\alpha\theta}\sigma_{\alpha\alpha}\right)\right]_{\hat{\alpha},\hat{\theta}} \quad (2.21)$$

By some manipulation to the mathematical derivations, we can get the explicit forms of α and θ as follows

$$\hat{\hat{\alpha}} = \hat{\alpha} + 1/\frac{m}{\hat{\alpha}^2} + \hat{\theta}(\hat{\theta}-1)\, ln(x_m)^2 x_m^{2\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-2} + \hat{\theta}\, ln(x_m)^2 x_m^{\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-1} - (\hat{\theta}-1)\sum_{i=1}^{m}\frac{ln(x_i)^2 x_i^{\hat{\alpha}}}{\left(1+x_i^{\hat{\alpha}}\right)^2}\Bigg[\frac{\hat{a}-1}{\hat{\alpha}} - b + \frac{1}{2}\left\{\frac{x_m^{\hat{\alpha}} ln(x_m)\, ln(1+x_m^{\hat{\alpha}})(1+x_m^{\hat{\alpha}})^{\hat{\theta}-1}(2+\hat{\theta}\, ln(1+x_m^{\hat{\alpha}}))}{-\frac{m}{\hat{\theta}^2} - ln(1+x_m^{\hat{\alpha}})^2(1+x_m^{\hat{\alpha}})^{\hat{\theta}}} + \left(\frac{m}{\hat{\alpha}^3} + (\hat{\theta}-1)\left[\sum_{i=1}^{m}\frac{ln(x_i)^3 x_i^{\hat{\alpha}}}{\left(1+x_i^{\hat{\alpha}}\right)^2} + \hat{\theta}\, ln(x_m)^3\left(\frac{-x_m^{\hat{\alpha}}(1+x_m^{\hat{\alpha}})^{\hat{\theta}-1}}{(\hat{\theta}-1)} - 3x_m^{2\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-2} - (\hat{\theta}-2)x_m^{3\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-3}\right)\right]\right)\left(\frac{m}{\hat{\alpha}^2} + \hat{\theta}(\hat{\theta}-1)\, ln(x_m)^2 x_m^{2\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-2} + \hat{\theta}\, ln(x_m)^2 x_m^{\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-1} - (\hat{\theta}-1)\sum_{i=1}^{m}\frac{ln(x_i)^2 x_i^{\hat{\alpha}}}{\left(1+x_i^{\hat{\alpha}}\right)^2}\right)^{-1}\right\}\Bigg] \quad (2.22)$$

and

$$\hat{\hat{\theta}} = \hat{\theta} + \frac{\hat{\theta}^2}{m + \hat{\theta}^2\, ln(1+x_m^{\hat{\alpha}})^2(1+x_m^{\hat{\alpha}})^{\hat{\theta}}}\left\{\left(\frac{c-1}{\hat{\theta}} - d\right) + \frac{1}{2}\left[\left(\sum_{i=1}^{m}\frac{ln(x_i)^2 x_i^{\hat{\alpha}}}{\left(1+x_i^{\hat{\alpha}}\right)^2} - ln(x_m)^2 x_m^{\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-1}\left[1 + \hat{\theta}\, ln(1+x_m^{\hat{\alpha}})\right] - \hat{\theta}(\hat{\theta}-1)\, ln(x_m)^2 x_m^{2\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-2}\left[\frac{1}{\hat{\theta}} + \frac{1}{(\hat{\theta}-1)} + ln(1+x_m^{\hat{\alpha}})\right]\right)\left(\frac{m}{\hat{\alpha}^2} + \hat{\theta}(\hat{\theta}-1)\, ln(x_m)^2 x_m^{2\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-2} + \hat{\theta}\, ln(x_m)^2 x_m^{\hat{\alpha}}\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}-1} - (\hat{\theta}-1)\sum_{i=1}^{m}\frac{ln(x_i)^2 x_i^{\hat{\alpha}}}{\left(1+x_i^{\hat{\alpha}}\right)^2}\right)^{-1} + \frac{2m - \hat{\theta}^3\, ln(1+x_m^{\hat{\alpha}})^3(1+x_m^{\hat{\alpha}})^{\hat{\theta}}}{\hat{\theta}\left(m + \hat{\theta}^2\, ln(1+x_m^{\hat{\alpha}})^2(1+x_m^{\hat{\alpha}})^{\hat{\theta}}\right)}\right]\right\} \quad (2.23)$$

where $\hat{\alpha}$ and $\hat{\theta}$ are MLEs of α and θ, respectively.

3. Prediction of Future Record Values

This section is devoted to studying the classical and Bayesian predictions of future record values based on a sample of record values from gpw distribution.

3.1. Classical prediction

Let we have m observed upper record values from gpw distribution are $X_{U(i)} = x_i; 1 \leq i \leq m$. Based on this sample, we intend to predict the future *n-th* upper record value $X_{U(n)}; n > m$. The joint predictive likelihood function of $X_{U(n)} = x_n$, is given by

$$L(x_n;\alpha,\theta,x) = \frac{[ln\,\bar{F}(x_m;\alpha,\theta) - ln\,\bar{F}(x_n;\alpha,\theta)]^{n-m-1}}{\Gamma(n-m)} \prod_{i=1}^{m} \frac{f(x_i;\alpha,\theta)}{\bar{F}(x_i;\alpha,\theta)} f(x_n;\alpha,\theta) \qquad (3.1)$$

Accordingly, we get the predictive likelihood function of gpw distribution as follow

$$L(x_n;\alpha,\theta,x) \propto (\alpha\theta)^{m+1}\left[(1+x_n^{\alpha})^{\theta} - (1+x_m^{\alpha})^{\theta}\right]^{n-m-1} x_n^{\alpha-1}(1 + x_n^{\alpha})^{\theta-1} e^{1-(1+x_n^{\alpha})^{\theta}} \prod_{i=1}^{m} x_i^{\alpha-1}(1+x_i^{\alpha})^{\theta-1} \qquad (3.2)$$

By taking the logarithm of Eq. (3.2) we get

$$lnL(x_n;\alpha,\theta,x) = (m+1)\,ln(\alpha\theta) + (n-m-1)\,ln\left[(1+x_n^{\alpha})^{\theta} - (1+x_m^{\alpha})^{\theta}\right] + (\alpha-1)\,ln(x_n) + (\theta-1)\,ln(1+x_n^{\alpha}) + 1 - (1+x_n^{\alpha})^{\theta} + (\alpha-1)\sum_{i=1}^{m} ln(x_i) + (\theta-1)\sum_{i=1}^{m} ln(1+x_i^{\alpha}) \qquad (3.3)$$

By differentiation of the Eq. (3.3) with respect to α, θ and x_n, we obtain the predictive likelihood equations of gpw distribution as follow

$$\frac{m+1}{\theta} + (n-m-1)\frac{ln(1+x_n^{\alpha})(1+x_n^{\alpha})^{\theta} - ln(1+x_m^{\alpha})(1+x_m^{\alpha})^{\theta}}{(1+x_n^{\alpha})^{\theta} - (1+x_m^{\alpha})^{\theta}} + ln(1+x_n^{\alpha})[1-(1+x_n^{\alpha})^{\theta}] + \sum_{i=1}^{m} ln(1+x_i^{\alpha}) = 0 \qquad (3.4)$$

$$\frac{m+1}{\alpha} + (n-m-1)\frac{\theta x_n^{\alpha} ln x_n (1+x_n^{\alpha})^{\theta-1} - \theta x_m^{\alpha} ln x_m (1+x_m^{\alpha})^{\theta-1}}{(1+x_n^{\alpha})^{\theta} - (1+x_m^{\alpha})^{\theta}} + ln(x_n) + \frac{(\theta-1)x_n^{\alpha} ln x_n}{(1+x_n^{\alpha})} - \theta x_n^{\alpha} ln x_n (1+x_n^{\alpha})^{\theta-1} + \sum_{i=1}^{m} ln(x_i) + (\theta-1)\sum_{i=1}^{m}\frac{x_i^{\alpha} ln x_i}{(1+x_i^{\alpha})} = 0 \qquad (3.5)$$

$$\frac{\alpha-1}{x_n} + (n-m-1)\frac{\alpha\theta x_n^{\alpha-1}(1+x_n^{\alpha})^{\theta-1}}{(1+x_n^{\alpha})^{\theta} - (1+x_m^{\alpha})^{\theta}} - \frac{\alpha(\theta-1)x_n^{\alpha-1}}{1+x_n^{\alpha}} - \alpha\theta x_n^{\alpha-1}(1+x_n^{\alpha})^{\theta-1} = 0 \qquad (3.6)$$

By solving the previous equations numerically, we get the point maximum likelihood prediction (MLP) $\hat{x}_n$ of the n^{th} upper record of gpw distribution.

3.2 Highest conditional prediction interval

A prediction interval of record value is a range of values that predicts a future record value, based on a sample of past record values. (Arnold et al., 2011) defined the conditional probability density function of $X_{U(n)}$ given $X_{U(m)}$, $1 \leq m < n$ as follows

$$f(x_n|x_m) = \frac{1}{\Gamma(n-m)}[\ln \bar{F}(x_m) - \ln \bar{F}(x_n)]^{n-m-1} \frac{f(x_n)}{\bar{F}(x_m)}, \quad x_m < x_n \tag{3.7}$$

Then, the conditional pdf of $X_{U(n)}$ given $X_{U(m)}$ of gpw distribution is

$$\hat{f}(x_n|x_m;\alpha,\theta) = \frac{\hat{\theta}\hat{\alpha}\left[\left(1+x_n^{\hat{\alpha}}\right)^{\hat{\theta}} - \left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}}\right]^{n-m-1}}{\Gamma(n-m)} \frac{x_n^{\hat{\alpha}-1}\left(1+x_n^{\hat{\alpha}}\right)^{\hat{\theta}-1} e^{-\left(1+x_n^{\hat{\alpha}}\right)^{\hat{\theta}}}}{e^{-\left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}}}},$$

$$x_m < x_n < \infty \tag{3.8}$$

where $\hat{\alpha}$ and $\hat{\theta}$ are maximum likelihood estimates of the parameters α and θ. Then, the $100(1-\tau)\%$ highest conditional density prediction limits for $X_{U(n)}$ can be gotten from following equations:

$$L_{HCD} = (1+l)x_m \qquad and \qquad U_{HCD} = (1+h)x_m \tag{3.9}$$

where l and h are the simultaneous solution of the two following equations:

$$\int_{(1+l)x_m}^{(1+h)x_m} \hat{f}(x_n|x_m;\alpha,\theta)\,dx_n = 1-\tau \tag{3.10}$$

and

$$\hat{f}((1+l)x_m|x_m)/\hat{f}((1+h)x_m|x_m) = 1 \tag{3.11}$$

By solving the equations Eqs. (3.10) and (3.11) numerically, we get the values of l and h, and then the prediction intervals of $X_{U(n)}$ are obtained from Eq. (3.9).

3.3 Bayesian prediction method

Let $\{X_{U(i)} = x_i; i = 1, \ldots, m\}$ are the first m observed upper record values. The Bayesian predictive density function of $X_{U(n)}, n > m$ given observed past records $\underline{X}$, is

$$g(x_n|\underline{x}) = \int_{\theta,\alpha} f(x_n|x_m;\alpha,\theta)\,\pi(\alpha,\theta|\underline{x})\,d\alpha,\theta \tag{3.13}$$

where $\pi(\alpha,\theta|\underline{x})$ and $f(x_n|x_m;\alpha,\theta)$ is the functions in Eqs. (2.13) and (3.7) respectively. Then, the predictive density function of x_n given the past records $\underline{x}$ from gpw distribution is

$$g(x_n|\underline{x}) = \int_{\alpha}\int_{\theta} \frac{1}{\mathcal{R}\Gamma(n-m)} \alpha^{a+m}\theta^{c+m} x_n^{\hat{\alpha}-1}\left(1+x_n^{\hat{\alpha}}\right)^{\hat{\theta}-1} e^{-b\alpha-d\theta-(1+x_n^{\hat{\alpha}})^{\hat{\theta}}}\left[\left(1+x_n^{\hat{\alpha}}\right)^{\hat{\theta}} - \left(1+x_m^{\hat{\alpha}}\right)^{\hat{\theta}}\right]^{n-m-1} \prod_{i=1}^{m} x_i^{\alpha-1}\left(1+x_i^{\alpha}\right)^{\theta-1} d\theta d\alpha \tag{3.14}$$

The Bayesian point prediction of the n-th upper record value from gpw distribution based on (SEL) function is the expected predictive density function in Eq. (3.14), as follow

$$\hat{x}_n = E(x_n|\underline{x}) = \int_{x_m}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{\mathcal{R}\Gamma(n-m)} \alpha^{a+m}\theta^{c+m} x_n^{\alpha}(1+x_n^{\alpha})^{\theta-1} e^{-b\alpha-d\theta-(1+x_n^{\alpha})^{\theta}}\left[(1+x_n^{\alpha})^{\theta} - (1+x_m^{\alpha})^{\theta}\right]^{n-m-1} \prod_{i=1}^{m} x_i^{\alpha-1}(1+x_i^{\alpha})^{\theta-1} d\alpha d\theta dx_n \tag{3.15}$$

Bayesian prediction bounds for $X_{U(n)}$ can be derived by computing $P\left(X_{U(n)} \geq q|\underline{x}\right)$, as follows

$$P\left(X_{U(n)} \geq q|\underline{x}\right) = \int_{g}^{\infty} g(x_n|\underline{x})\,dx_n \tag{3.16}$$

where, q is a positive value.

Also, the interval bounds of the Bayesian predictive $P\left[L_B < X_{U(n)} < U_B\right] = \tau$, can be acquired by solving the following equations:

$$P\left(X_{U(n)} > L_B|\underline{x}\right) = \frac{(1+\tau)}{2} \tag{3.17}$$

and

$$P\left(X_{U(n)} > U_B|\underline{x}\right) = \frac{(1-\tau)}{2} \tag{3.18}$$

where L_B and U_B are the lower and upper Bayesian predictive bounds, respectively. To obtain L_B and U_B, a numerical integration may be needed to solve equations (3.17), (3.18) due to the difficulty of obtaining their solutions analytically.

4. APPLICATION TO REAL DATA

In this section, for illustration purpose we applied our proposed procedures to a real data set. We consider the real data set of total annual rainfall (in inches) during month of January from 1917 to 1969 recorded at Los Angeles Civic Center (see the website of Los Angeles Almanac: http://www.laalmanac.com/weather/we13.php) that are: 0.5, 0.96, 0.5, 3.28, 4.64, 1.76, 0.36, 0.2, 3.06, 1.09, 0.02, 1.2, 6.55, 3.9, 2.94, 8.46, 3.22, 2.91, 0.51, 1.99, 1.63, 2.96, 4.33, 2.21, 0.59, 7.98, 0.96, 0.04, 0.11, 0.38, 1.5, 2.43, 2.57, 2.8, 10.03, 1.08, 4.6, 4.3, 8.39, 4.41, 2.08, 1.24, 2.94, 1.28, 2.56, 0.52, 1.43, 0.84, 0.96, 5.93, 0.9, 14.94. The descriptive statistics for these data are display in Table (1). This indicates that the distribution of data has a heavy right tail.

Table (1) Descriptive Statistics for the real data

mean	median	Mode	Variance	Skewness	Kurtosis	min	Max	N
2.824	2.08	1	8.362	2.0145	4.793	0.02	14.94	53

We can check how well the gpw distribution fits our dataset by using the Kolmogorov-Smirnov (K-S) goodness of fit test. The test statistic is KS-D = 0.0948 with P-value = 0.7273 (evaluated by the MLEs of the model parameters $\hat{\alpha} = 1.2803$ and $\hat{\theta} = 0.4813$). Thus, the gpw model provides a good fit for this data. This can be also deduced through the plots of Quantile-Quantile(Q-Q) and the corresponding histogram along with the fitted gpw density function in Figure 1.

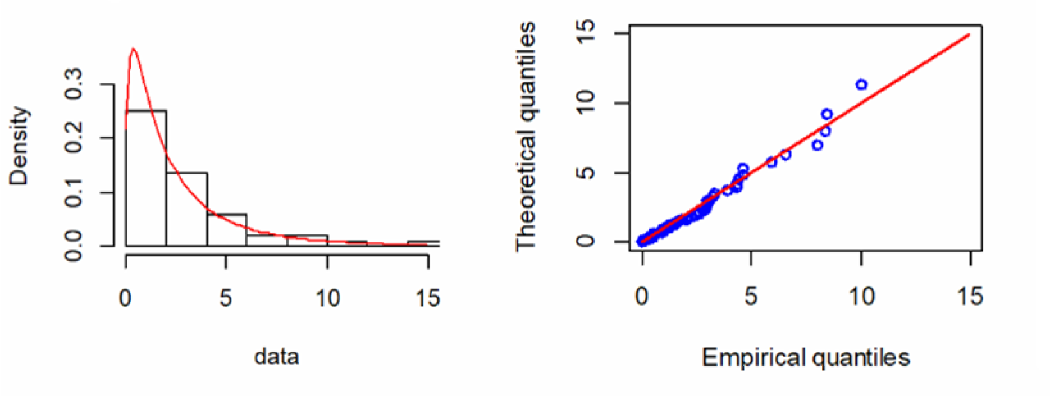

Fig 1. The histogram and theoretical density (left figure); and Quantile-Quantile(Q-Q) plot (right figure) for rainfall data using MLEs.

In our dataset, we observe 8 upper record values are: 0.5, 0.96, 3.28, 4.64, 6.55, 8.46, 10.03, 14.94. Based on these upper record values, the maximum likelihood estimates of α and θ from Eqs. (2.4, 2.5), are respectively, $\hat{\alpha}_{ML} = 0.825$ and $\hat{\theta}_{ML} = 0.932$. To show that the likelihood equations have a unique solution, the profile log-likelihood function of α and θ are provided in Fig. 2. Also, Bayes estimates of α and θ under the SE loss function with non-informative prior ($a = b = c = d = 0$) by using Lindley approximation in Eqs. (2.22, 2.23), are $\hat{\alpha}_{BS} = 0.764$ and $\hat{\theta}_{BS} = 0.869$. To compare the performance of these estimates, the maximum likelihood and Bayes estimates are used to plot the empirical and fitted cdf in Figure 3. These plots have shown that the Bayes estimates provide a better fit than the maximum likelihood estimates.

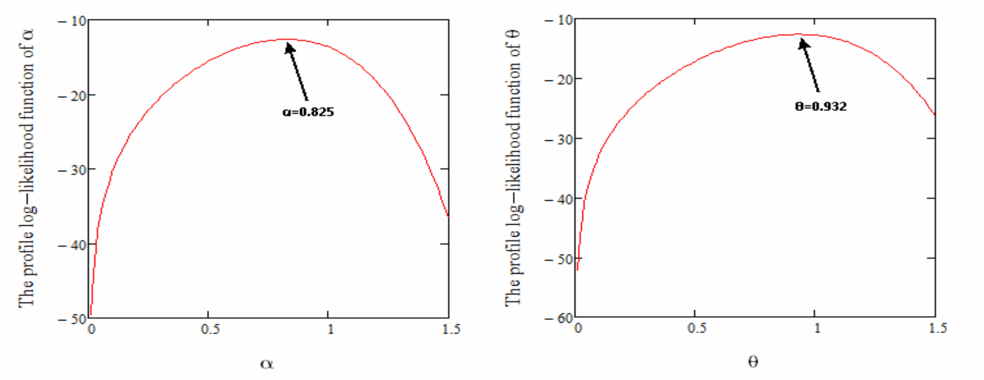

Fig 2. The profile log-likelihood function of α and θ

In our recorded values sample of rainfall, we assume that the last two events have not yet been recorded and we need to predict them. This means the first 6 records will be used to predict the future eighth record value of precipitation. The MLP for the eighth upper record value from Eqs. (3.4: 3.6) is (9.669) and the highest conditional interval with 95% confidence level from Eq. (3.9) is (8.566, 12.079). Also, the Bayesian point prediction for the eighth upper record value from Eq. (3.15) is (15.286) and the 95% prediction interval from Eqs. (3.17, 3.18) is (13.448, 16.953). We can notice that Bayes' prediction is closer to the true recorded value than the maximum likelihood prediction. Also, it should be noted that the true value of the eighth record value falls in the Bayesian prediction interval.

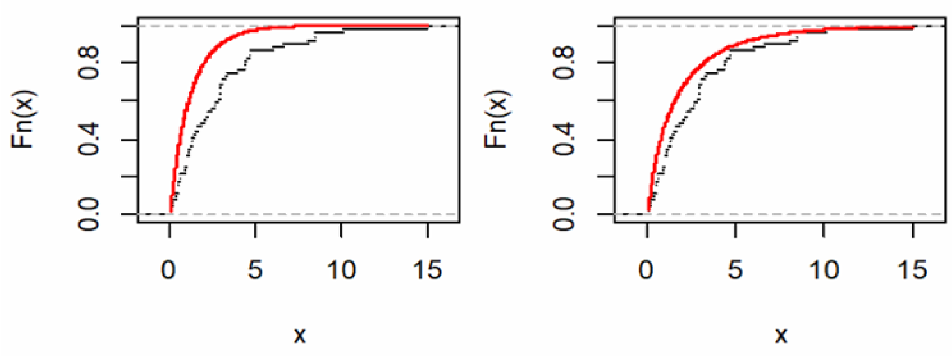

Fig. 3: Empirical and fitted cdf for rainfall data using MLEs (left panel); Empirical and fitted cdf for rainfall data using Bayes estimates (right panel).

5. NUMERICAL EXAMPLE

In this section, the simulation study is carried out to assess the performance of the proposed methods and for comparing between them practically. Table 2, displays samples of upper record values generated from gpw distribution by using Eq. (1.5) for parameters values $\alpha = 0.8, 1.3$ and $\theta = 0.5, 0.8, 1.3, 2$. The Bayesian inference are made under the squared error loss function depending on non-informative $(a = b = c = d = 0)$ and informative priors ($a, b, c, d > 0$). To assess the performance of the estimates we use the percentage errors $PE = \frac{|estimate\ value - exact\ value|}{|exact\ value|} \times 100$. Tables 3 and 4, display the Bayesian and non-Bayesian estimates of the parameters α and θ, along with their (PE). Tables 5 and 6, display the Bayesian and non Bayesian predictions along with their PE.

Table 2: *Samples of upper record values from gpw distribution for various parameters values* $\alpha = 0.8,1.3$ *and* $\theta = 0.5,0.8,1.3,1.5$

α	θ	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
0.8	*0.5*	0.54	4.73	11.639	24.505	26.85	36.746	40.606	45.467	98.496	133.632	158.781	198.419
	0.8	0.639	1.518	4.077	4.10	5.108	14.585	16.262	19.364	24.045	28.962	33.416	40.795
	1.3	0.479	0.657	0.847	1.868	3.23	3.678	4.306	5.027	5.7	6.5	7.617	7.978
	2	0.045	0.578	0.759	1.221	1.569	1.574	1.832	1.965	2.156	2.592	2.891	2.961
1.3	*0.5*	0.712	7.45	8.831	10.814	11.678	12.978	14.32	16.796	19.287	22.001	24.799	28.527
	0.8	0.761	0.872	2.97	5.342	5.356	6.076	6.292	7.138	7.981	8.8	9.648	11.181
	1.3	0.658	1.032	1.273	1.64	1.672	1.78	1.84	2.245	2.675	2.853	3.021	3.201
	2	0.358	0.647	0.791	0.912	1.147	1.31	1.353	1.412	1.489	1.677	1.882	1.911

Table 3: maximum likelihood and Bayes estimates of α and θ and percentage errors (in the parentheses), for $\alpha = 0.8$

θ	M	MLEs		Non-informative Bayes		Informative Bayes	
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}_{LB}$	$\hat{\theta}_{LB}$	$\hat{\alpha}_{LB}$	$\hat{\theta}_{LB}$
0.5	8	0.97 (21.294)	0.602(20.387)	0.909(13.614)	0.583(16.558)	0.893(11.624)	0.563(12.661)
	9	0.87(8.743)	0.571(14.188)	0.82(2.452)	0.555(10.945)	0.811(1.372)	0.537(7.455)
	10	0.857(7.085)	0.565(13.06)	0.813(1.618)	0.551(10.152)	0.806(0.769)	0.536(7.179)
	12	0.852(6.508)	0.564(12.762)	0.817(2.187)	0.552(10.334)	0.813(1.57)	0.541(8.11)
0.8	8	0.951(18.902)	0.754(5.807)	0.884(10.475)	0.701(12.324)	0.886(8.29)	0.724(9.458)
	9	0.921(15.113)	0.763(4.646)	0.865(8.083)	0.717(10.343)	0.852(6.526)	0.736(7.969)
	10	0.900(12.562)	0.770(3.76)	0.852(6.555)	0.730(8.795)	0.843(5.377)	0.746(6.79)
	12	0.856(7.029)	0.789(1.413)	0.820(2.494)	0.756(5.514)	0.815(1.815)	0.768(4.012)
1.3	8	0.876(9.522)	1.337(2.821)	0.802(0.282)	1.232(5.248)	0.809(1.134)	1.25(3.869)
	9	0.872(8.97)	1.334(2.64)	0.809(1.123)	1.244(4.302)	0.815(1.819)	1.259(3.172)
	10	0.851(6.375)	1.341(3.165)	0.798(0.288)	1.261(2.965)	0.803(0.362)	1.274(1.984)
	12	0.84(4.94)	1.340(3.077)	0.799(0.180)	1.278(1.707)	0.802(0.310)	1.287(0.974)
2	8	0.889(11.164)	2.135(6.736)	0.802(0.285)	2.022(1.12)	0.763(4.642)	1.971(1.436)
	9	0.886(10.763)	2.127(6.353)	0.809(1.07)	2.026(1.297)	0.779(2.633)	1.984(0.797)
	10	0.784(2.046)	2.12(5.992)	0.723(9.666)	2.028(1.403)	0.73(8.805)	1.996(0.2)
	12	0.806(0.748)	2.103(5.154)	0.756(5.498)	2.028(1.378)	0.76(4.972)	2.004(0.203)

Table 4: maximum likelihood and Bayes estimates of α and θ and percentage errors (in the parentheses), for $\alpha = 1.3$

θ	M	MLEs		Non-informative Bayes		Informative Bayes	
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}_{LB}$	$\hat{\theta}_{LB}$	$\hat{\alpha}_{LB}$	$\hat{\theta}_{LB}$
0.5	8	1.433(10.23)	0.560(12.01)	1.345(3.497)	0.525(5.039)	1.324(1.871)	0.519(3.731)
	9	1.428(9.822)	0.559(11.86)	1.352(3.986)	0.529(5.808)	1.334(2.64)	0.524(4.712)
	10	1.422(9.359)	0.558(11.601)	1.355(4.227)	0.531(6.273)	1.34(3.095)	0.527(5.341)
	12	1.407(8.193)	0.553(10.556)	1.353(4.089)	0.531(6.294)	1.342(3.257)	0.528(5.599)
0.8	8	1.421(9.333)	0.778(2.744)	1.326(1.968)	0.727(9.112)	1.302(0.134)	0.708(11.544)
	9	1.412(8.597)	0.778(2.720)	1.33(2.293)	0.734(8.235)	1.311(0.823)	0.718(10.263)
	10	1.406(8.164)	0.778(2.704)	1.335(2.672)	0.74(7.547)	1.319(1.454)	0.726(9.27)
	12	1.337(2.872)	0.786(1.735)	1.283(1.328)	0.755(5.681)	1.273(2.062)	0.744(7.042)
1.3	8	2.139(64.551)	1.149(11.578)	1.969(51.463)	1.073(17.484)	1.837(41.27)	1.016(21.837)
	9	1.672(28.583)	1.248(4.023)	1.553(19.459)	1.175(9.594)	1.499(15.325)	1.117(14.048)
	10	1.654(27.245)	1.25(3.861)	1.552(19.418)	1.186(8.764)	1.456(12.025)	1.155(11.161)
	12	1.609(23.761)	1.261(2.977)	1.531(17.807)	1.21(6.919)	1.467(12.878)	1.186(8.76)
2	8	1.696(30.441)	2.133(6.661)	1.532(17.867)	2.02(0.988)	1.169(10.085)	1.836(8.2)
	9	1.678(20.101)	2.124(6.203)	1.532(17.836)	2.022(1.087)	1.263(2.858)	1.871(6.451)
	10	1.418(9.055)	2.122(6.11)	1.308(0.6)	2.029(1.471)	1.174(9.692)	1.901(4.953)
	12	1.316(1.207)	2.116(5.822)	1.234(5.105)	2.04(1.981)	1.162(10.639)	1.944(2.81)

Table 5: the classical and Bayesian predictions for the future sth upper record value and the PE (in the parentheses) for $\alpha = 0.8$

θ	m, s	Non-Bayesian predictions		Non-informative Bayes		Informative Bayes	
		$\hat{x}_s$	L_{HCD}, U_{HCD}	$\tilde{x}_s$	L_B, U_B	$\tilde{x}_s$	L_B, U_B
0.5	8, 10	52.277 (60.88)	46.035, 64.875	77.399(42.08)	98.733, 141.48	72.209(45.965)	98.507, 135.823
	9,11	114.944(27.609)	99.727, 139.949	174.882(10.141)	119.68,159.54	168.309(6.001)	119.692,159.54
	10, 12	154.877 (21.945)	134.202, 192.73	224.368(13.078)	120.10,199.63	217(9.586)	120.092,199.63
0.8	8,10	21.983(24.097)	19.384, 28.161	32.552(12.397)	38.949,48.105	31.522(8.838)	38.904,47.956
	9,11	27.082(18.954)	24.045, 43.351	37.253(11.481)	34.615, 48.832	36.483(9.18)	34.616, 48.771
	10, 12	32.377(20.635)	28.973, 42.009	42.308(3.709)	26.11, 49.32	41.704(2.227)	26.134, 49.3
1.3	8,10	5.506(15.293)	5.027, 7.272	7.271(11.86)	5.779, 7.533	7.182(10.496)	5.781, 7.532
	9,11	6.195(18.669)	5.7, 8.503	7.723(1.398)	6.391,8.082	7.674(0.744)	6.332, 8.009
	10, 12	7.026(11.927)	6.5, 9.438	8.439(5.777)	7.464, 8.359	8.404(5.344)	7.416, 8.306
2	8,10	2.103(18.882)	1.965, 2.717	2.51(3.161)	2.504, 2.806	2.522(2.715)	2.507, 2.819
	9,11	2.293(20.677)	2.156, 2.892	2.764(4.379)	2.582, 2.903	2.78(3.84)	2.582, 2.913
	10, 12	2.763(6.683)	2.592, 3.516	2.823(4.677)	2.733, 3.213	2.824(4.64)	2.733, 3.216

Table 6: the classical and Bayesian predictions for the future sth upper record value and the PE (in the parentheses) for $\alpha = 1.3$

θ	m, s	Non-Bayesian predictions		Non-informative Bayes		Informative Bayes	
		$\hat{x}_s$	L_{HCD}, U_{HCD}	$\tilde{x}_s$	L_B, U_B	$\tilde{x}_s$	L_B, U_B
0.5	8, 10	18.615(15.392)	16.796, 27.736	24.478(11.26)	19.771,27.405	24.223(10.097)	19.774,27.398
	9,11	21.222(14.425)	19.287, 28.006	26.645(7.445)	19.062, 27.63	26.469(6.735)	19.049, 27.628
	10, 12	21.481(6.405)	19.744, 29.63	29.248(2.529)	20.034, 19.868	29.115(2.06)	20.034, 31.595
0.8	8,10	7.756(11.865)	7.138, 10.337	9.587(8.946)	7.817, 10.50	9.543(8.439)	7.819, 10.499
	9,11	8.617(10.689)	7.981, 11.463	10.262(6.361)	8.841, 10.678	10.236(6.094)	8.841, 10.678
	10, 12	9.448(15.502)	8.8, 12.299	10.957(2.001)	9.853, 10.799	10.941(2.144)	9.854, 10.80
1.3	8,10	2.346(17.771)	2.245, 2.743	2.797(1.962)	2.33, 3.218	2.81(1.491)	2.179, 1.975
	9,11	2.807(7.092)	2.675, 3.339	3.205(6.081)	2.536, 3.634	3.216(6.467)	2.539, 3.654
	10, 12	2.982(6.836)	2.853, 3.499	3.336(4.207)	2.505, 3.727	3.346(4.527)	2.505, 3.745
2	8,10	1.466(12.565)	1.412, 1.68	1.72(2.59)	1.704, 1.832	1.738(3.65)	1.711, 1.863
	9,11	1.541(18.112)	1489, 1.745	1.759(6.555)	1.749, 1.856	1.773(5.796)	1.754, 1.88
	10, 12	1.74(8.929)	1.677, 1.99	1.93(1.011)	1.59, 2.039	1.943(1.667)	1.588, 2.06

6. RESULTS AND DISCUSSION

In view of Tables 3 and 4, we notice that for both estimated parameters, the PE of the Bayes estimates either with informative or non-informative priors are smaller than the PE of ML estimates. However, the PE of Bayes estimates under informative prior are smaller than the PE of Bayes estimates under non-informative. Further, the PE of all estimators decreases with increasing sample size. Besides, Tables 5 and 6 show that the Bayes point prediction either under informative or non-informative has smaller PE than the maximum likelihood point prediction as well the Bayesian prediction has shorter interval compared to highest conditional prediction. However, all predictions are improved with increasing sample size according to EP. Finally, the Bayesian method is superior to the ML method in both the estimation and prediction of the gpw distribution for the record values.

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