Estimating and Prediction for Alpha-Power Pareto distribution Based on Progressive Type-II Censoring Scheme

Refaey, R. M.

Statistics Department, Faculty of Commerce,

AL-Azhar University (Girls'Branch), Cairo, Egypt

Abstract

In Statistical theory, inclusion of an additional parameter to standard distributions is a usual practice. Mahdavi and Kundu (2017) presented a method, called alpha power transformation, for including an extra parameter in continuous distribution. Basically, the idea was introduced to incorporate skewness to the baseline distribution. In this paper, the parameters of the alpha-power Pareto distribution, reliability and hazard rate functions are estimated under progressive censoring Type-II scheme with random removal. The model parameters are estimated using the maximum likelihood estimation method. Further, the asymptotic confidence intervals for the model parameters are discussed. Maximum likelihood prediction (point and bounds) is considered for future order statistics under progressive Type-II censored informative samples. Numerical study is given and some interesting comparisons are presented to illustrate the theoretical results. Moreover, the results are applied to real data sets.

Keywords: *alpha-power Pareto distribution; progressive Type-II censored samples; maximum likelihood method; asymptotic Fisher information matrix; two-sample prediction; maximum likelihood prediction.*

1. Introduction

Life-testing and reliability experiments contain many situations where units are removed or lost from the test before failure. In many scenarios, the removal of units before failure is very often procedure due to limitations of time and cost associated with the experiment. The data of such tests or experiments are called censored data. There are different types of censoring schemes which include right, left, interval censoring, single or multiple censoring and Type-I or Type-II censoring. The most common censoring schemes are Type-I and Type-II censoring, but the conventional Type-I and Type-II censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. In Type-I censoring a life test is conducted for a fixed-time period while in Type-II censoring an experiment terminates when a prescribed number of units fail. Type-I and Type-II censoring schemes have probably found the most extensive applications in these situations. For example, units may break accidently in an industrial experiment, individuals may drop out of the study in a clinical trial, or they have to be terminated early due to lack of funds. Also, some products have to be withdrawn for more thorough inspection or saved for use as test specimens in other studies. One drawback of these schemes is that live units can be removed only at the end of the experiment. However in much life testing experiment, it is desired to withdraw live units from the experiment at time points other than the final termination point of the test. For this reason, the progressive censoring possesses such flexibility and thus allows in between removals of units as well. Different inferential procedures based on progressively censored samples have been discussed by several authors, including Balakrishnan (2007) and Balakrishnan and Aggarwala (2000). The most popular one is known as the progressive Type-II censoring scheme and it can be briefly described as follows:

Considering n identical units are put to test and the lifetime distribution of the n units is denoted by $X_1, X_2, \dots, X_n$.The integer $m(< n)$ is fixed at the beginning of the experiment and $R_1, R_2, \dots, R_m$ are m pre-fixed integers satisfying $R_1 + R_2 + \cdots + R_m + m = n$. At the time of the first failure $X_{1:m:n}$, R_1 units are chosen randomly from the remaining $n - 1$ units and they are removed from the experiment. Similarly at the time of the second failure $X_{2:m:n}$, R_2 of the remaining $n - R_1 - 2$ units are removed from the test and so on. Finally, when the m-th failure is observed the experiment is terminated and the remaining surviving units R_m with $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ are removed, see Figure 1. Here $(R_1, R_2, \dots, R_m)$ is known as the censoring scheme and it is prefixed before the experiment starts. [For more details about the progressive Type-II censoring scheme, see, Balakrishnan and Aggarwala (2000), Balakrishnan and Cramer (2014), Almetwally *et al*. (2018), Almetwally and Almongy (2018), Karakoca and Pekgör (2019) and Alshenawy *et al*. (2020)].

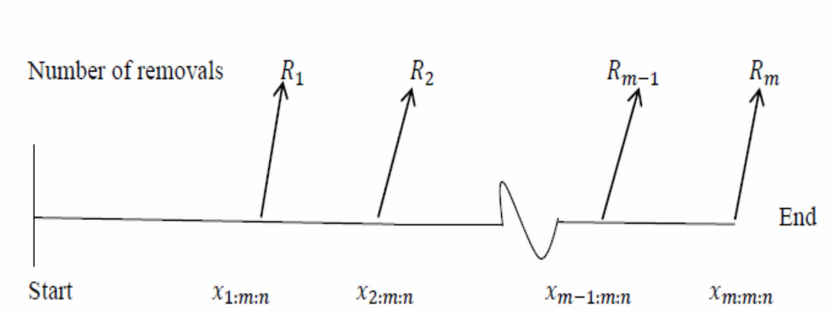

Figure 1: The plot of steps of progressive Type-II censoring scheme

Pareto distribution is a well-known distribution used to model heavy tailed phenomena. It has many applications in actuarial science, survival analysis, economics, life testing, hydrology, finance, telecommunication, reliability analysis, physics and engineering. Pareto distribution is successfully used by Philbrick (1985) for projection of losses in an insurance company, real state and liability experience of hospitals. Farshchian (2010) applied Pareto distribution to model sea clutter intensity returns. Levy (2003) used Pareto distribution for investigation of wealth in society. Castillo (1997) considered generalized form of Pareto distribution to model exceedances over a margin in flood control. Many types of Pareto distribution and its generalization are available in literature.

More recently, Mahdavi and Kundu (2017) presented a method, called alpha power transformation, for including an extra parameter in continuous distribution. Basically, the idea was introduced to incorporate skewness to the baseline distribution. Ihtisham *et al*. (2019) proposed more flexible distribution, by introducing an additional parameter to Basic Pareto distribution, to obtain an adequate fit. They studied the distribution, termed as *alpha power Pareto* (APP) distribution, which is derived using the alpha power transformation. Also, they presented mathematical and statistical properties of APP distribution along with application to two real lifetime data and provided graphical illustrations of the dimensions of APP distribution. Therefore, they estimated the parameters, using *maximum likelihood* (ML) method.

This paper deals with the APP distribution based on progressive Type-II censored samples since it has not been applied in all the previous literature. The objective of this paper is to obtain the ML estimators for the unknown parameters, *reliability function* (rf) and *hazard rate function* (hrf) for APP distribution based on progressive Type-II censored samples and confidence intervals for the parameters are constructed. Also, the ML prediction point and bounds are considered for future order statistics under progressive Type-II censored informative samples.

The rest of this paper is organized as follows: Section 2 presents a brief summary about the APP distribution. The statistical inference for APP distribution based on progressive Type-II censored samples is obtained in Section 3. A numerical study simulation and two real dataset are performed to investigate the precision of ML estimates in Section 4. Finally, some general conclusions are introduced in Section 5.

2. The Alpha Power Pareto Distribution

This section, presents a brief summary about the APP distribution based on progressive Type-II censored samples.

Ihtisham *et al*. (2019) constructed a distribution with two parameters
and presented mathematical statistical properties of APP distribution.
Assume a random variable X that follows the APP distribution, denoted by $X \sim$ APP (α, β). The *cumulative distribution function* (cdf) and *the probability density function* (pdf) for APP distribution are respectively written as:

$$F(x;\alpha,\beta) = \frac{\alpha^{1-x^{-\beta}}-1}{\alpha-1}, \qquad \alpha \neq 1, x > 1; (\beta > 0), \tag{1}$$

and

$$f(x;\alpha,\beta) = \beta ln(\alpha)\alpha^{1-x^{-\beta}}x^{-\beta-1}, \qquad \alpha \neq 1, x > 1; (\beta > 0), \tag{2}$$

where α is a shape parameter, β is a scale parameter and the pdf of APP distribution in (1) is decreasing function for $\alpha < 1$ and uni-modal and positively skewed for $\alpha <1$.
The rf of the APP distribution, based on (1), is given by

$$R(x;\alpha,\beta) = \frac{\alpha\left(1-\alpha^{-x^{-\beta}}\right)}{\alpha-1}, \qquad \alpha \neq 1, \qquad x > 1; (\beta > 0), \tag{3}$$

and

the hrf of the APP distribution based on (2) and (3), is given by

$$h(x;\alpha,\beta) = \frac{\beta ln(\alpha)\alpha^{-x^{-\beta}}x^{-\beta-1}}{1-\alpha^{-x^{-\beta}}}, \qquad \alpha \neq 1, \quad x > 1; (\beta > 0), \tag{4}$$

the hrf of APP distribution in (4) is increasing, decreasing shapes. Also, the rhrf, which is known by the dual of the hrf; describes the probability of an immediate past failure, given that the unit has already failed at time x, as opposed to the immediate future failure. The rhrf is given by

$$rh(x;\alpha,\beta) = \frac{(\alpha-1)\,\beta ln(\alpha)\alpha^{1-x^{-\beta}}x^{-\beta-1}}{\alpha^{1-x^{-\beta}}-1}, \qquad \alpha \neq 1, x > 1; (\beta > 0). \tag{5}$$

3. Maximum Likelihood Estimation and Predication

In this section, the estimation of the unknown parameters, rf and hrf for APP distribution under progressive Type-II censored samples is discussed in Subsection 3.1. The ML point and confidence interval are obtained in Subsection 3.2, 3.3, respectively. Also, two-sample prediction is considered in Subsection 3.4.

3.1 Maximum likelihood estimation

Let $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ denote a progressive Type-II censored sample obtained from APP (α, β) distribution. The *likelihood function* (LF) is given by

$$L(\underline{\theta}; \underline{x}) = C(n, m-1) \prod_{i=1}^{m} f(x_{(i)}; \underline{\theta}) \left[1 - F(x_{(i)}; \underline{\theta})\right]^{R_i}, \quad (6)$$

where $\underline{\theta} = (\alpha, \beta)'$, $\underline{x} = (x_{1:m:n}, x_{2:m:n}, \dots, x_{m:m:n})$ denotes an observed value of
$\underline{X} = (X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n})$ and
$C(n, m-1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \dots (n - R_1 - \cdots - R_{m-1} - m + 1)$, with $C(n, 0) = n$.

Then substituting (1) and (2) in (6) yields

$$L(\underline{\theta}; \underline{x}) = C(n, m-1) \beta^m \left(\frac{\ln(\alpha)}{\alpha - 1}\right)^m \alpha^{m - \sum_{i=1}^{m} x_i^{-\beta}} \prod_{i=1}^{m} \left[x_i^{-\beta-1}\right] \prod_{i=1}^{m} \left\{\frac{\alpha\left(1 - \alpha^{-x_i^{-\beta}}\right)}{\alpha - 1}\right\}^{R_i}. \quad (7)$$

3.1.1 Point estimation

The ML estimator of α and β are obtained by maximizing the logarithm of the likelihood function, denoted by ℓ which can be written in the form

$$\ell \propto m\ln(\beta) + m\ln\left(\frac{\ln(\alpha)}{\alpha-1}\right) + \left(m - \sum_{i=1}^{m} x_i^{-\beta}\right)\ln(\alpha) + (-\beta-1)\sum_{i=1}^{m}\ln(x_i)$$

$$+\sum_{i=1}^{m} R_i \ln\left[\frac{\alpha\left(1-\alpha^{-x_i^{-\beta}}\right)}{(\alpha-1)}\right], \qquad (8)$$

The first partial derivatives of the logarithm of the likelihood function with respect to α and β are given below:

$$\frac{\partial \ell}{\partial \alpha} = \frac{m(\alpha-1-\alpha\ln(\alpha))}{\alpha(\alpha-1)\ln(\alpha)} + \frac{\left(m-\sum_{i=1}^{m} x_i^{-\beta}\right)}{\alpha} +$$

(9)

$$\sum_{i=1}^{m} R_i \left[\frac{(\alpha-1)\left(\alpha\left(x_i^{-\beta}\left(\alpha^{-x_i^{-\beta}}-1\right)\right)+\left(1-\alpha^{-x_i^{-\beta}}\right)\right)-\alpha\left(1-\alpha^{-x_i^{-\beta}}\right)}{(\alpha-1)\alpha\left(1-\alpha^{-x_i^{-\beta}}\right)}\right],$$

and

$$\frac{\partial \ell}{\partial \beta} = \frac{m}{\beta} + \sum_{i=1}^{m} x_i^{-\beta}\ln(x_i)\ln(\alpha) - \sum_{i=1}^{m}\ln(x_i)$$

$$-\sum_{i=1}^{m} R_i \ln\left[\frac{\alpha^{-x_i^{-\beta}}\ln(\alpha)x_i^{-\beta}\ln(x_i)}{\left(1-\alpha^{-x_i^{-\beta}}\right)}\right], \qquad (10)$$

The ML estimators are obtained by setting (9)-(10) to zeros. The system of non-linear equations can be solved numerically using Newton-Raphson method, to obtain the ML estimators of $\hat{\alpha}$ and $\hat{\beta}$. Depending on the invariance property of the ML estimators, then, the ML estimators of the rf, hrf and rhrf are obtained by replacing the parameters α and β in (3), (4) and (5), respectively by their ML estimators.

Hence, for a given value of t, the ML estimators of $R(t), h(t)$ and $rh(t)$ are as follows:

$$\hat{R}(t_0) = \frac{\hat{\alpha}\left(1-\hat{\alpha}^{-x^{-\beta}}\right)}{\hat{\alpha}-1},$$

$$\hat{\alpha} \neq 1, t_0 > 1; (\beta > 0), \qquad (11)$$

$$\hat{h}(t_0) = \frac{\hat{\beta} \ln(\hat{\alpha}) \hat{\alpha}^{-x^{-\hat{\beta}}} x^{-\hat{\beta}-1}}{1-\hat{\alpha}^{-x^{-\hat{\beta}}}}$$

$$, \quad \hat{\alpha} \neq 1, t_0 > 1; (\beta > 0), \qquad (12)$$

and

$$\widehat{rh}(t_0) = \frac{(\hat{\alpha}-1)\, \hat{\beta} \ln(\hat{\alpha}) \hat{\alpha}^{1-x^{-\hat{\beta}}} x^{-\hat{\beta}-1}}{\hat{\alpha}^{1-x^{-\hat{\beta}}}-1},$$

$$\hat{\alpha} \neq 1, t_0 > 1; (\beta > 0). \qquad (13)$$

Remarks:

- If $R_i = 0, i = 1,2,\dots,m$ and n equals m, all the results obtained for progressive Type-II censored sample reduced to the complete sample case.
- If $R_i = 0, i = 1,2,\dots,m-1$ and $R_m = n-m$, then progressive Type-II censored sampling reduces to traditional Type-II censoring.

3.1.2 Confidence intervals

The *asymptotic variance covariance matrix* (AVCM) of the estimators $\hat{\alpha}$ and $\hat{\beta}$ are obtained depending on the *inverse asymptotic Fisher information matrix* (AFIM) using the second partial derivatives of the logarithm of the likelihood function.
The AFIM can be written as follows:

$\tilde{I} \approx -\left[\frac{\partial^2 l}{\partial\psi_i \partial\psi_j}\right]$,

where $i, j = 1, 2,$ (14)

where $\psi_1 = \alpha, and\ \psi_2 = \beta$.

For large sample size, the ML estimators under regularity conditions are consistent and asymptotically unbiased as well as asymptotically normally distributed. Therefore, the *asymptotic confidence interval* (ACI) for the parameters; ψ, can be obtained by

$P\left[-Z < \frac{\hat{\psi}_{iML} - \psi_i}{\sigma_{\hat{\psi}_{iML}}} < Z\right] = 1 - \tau$, where Z is the $100\left(1 - \frac{\tau}{2}\right)$ *th* standard normal percentile. The two-sided approximate $100(1 - \tau)\%$ the confidence intervals are

$L_{\psi_i} = \hat{\psi}_{iML} - Z_{\frac{\tau}{2}}\, \hat{\sigma}_{\hat{\psi}_{iML}}$,and $U_{\psi_i} = \hat{\psi}_{iML} + Z_{\frac{\tau}{2}}\, \hat{\sigma}_{\hat{\psi}_{iML}}$. (15)

Where $\hat{\sigma}_{\hat{\psi}_{iML}}$ is the standard deviation and $\hat{\psi}_{iML}\ is\ \hat{\alpha}, \hat{\beta}, \hat{R}(t_0)\ or\ \hat{h}(t_0)$ respectively.

3.2 Maximum likelihood prediction

In this subsection two-sample prediction is considered. ML point and interval prediction for a future observation based on progressive Type-II censored sample for APP distribution are discussed.

Assuming that $X_{(1)} \le X_{(2)} \le \cdots \le X_{(m)}$ are the first *m* ordered life times in a random sample of *n* components progressive Type-II censored sample whose failure times are identically distributed as a random variable *X* having the pdf for an item tested which is an informative sample, and that $T_{(1)}, T_{(2)}, \dots, T_{(r)}$ is a future independent random sample (of size *r*) from the same distribution. Our aim is to predict a statistic in the future sample based on the informative sample [see Kaminsky and Rhodin (1985), Ateya and Mohammed (2018) and Raqab *et al*. (2019)].

For the future sample of size *r*, let $T_{(s)}$ denotes the s^{th} order statistic, $1 \le s \le r$, then the pdf of $T_{(s)}$ is given by

$$f_{s:r}(t_{(s)};\underline{\omega}) = C(s)f(t_{(s)};\underline{\omega})[F(t_{(s)};\underline{\omega})]^{s-1}[1-F(t_{(s)};\underline{\omega})]^{r-s}, \quad t_{(s)} > 0$$

, (16)

$$\underline{\omega} = (\alpha,\beta), C_{s:r} = \frac{1}{B(s,r-s+1)}, s = 1,2,3,\dots,r,$$

using the binomial expansion theorem for $[1-F(t_{(s)};\underline{\omega})]^{r-s}$, yields

$$f_{s:r}(t_{(s)};\underline{\omega}) = C_{s:r}f(t_{(s)};\underline{\omega})\sum_{j=0}^{r-s}(-1)^j\binom{r-s}{j}[F(t_{(s)};\underline{\omega})]^{s+j-1}, \quad (17)$$

and substituting (1) and (2) into (17), then one can obtain the pdf of s^{th} order statistic for an item tested at accelerated conditions:

$$f_{s:r}(t_{(s)};\underline{\omega}) = C_{s:r}\beta ln(\alpha)\alpha^{1-t_{(s)}^{-\beta}}t_{(s)}^{-\beta-1}\sum_{j=0}^{r-s}(-1)^j\binom{r-s}{j}\left[\frac{\alpha^{1-t_{(s)}^{-\beta}}-1}{\alpha-1}\right]^{s+j-1}$$

$$\alpha \neq 1, t_{(s)} > 1; (\beta > 0). \quad (18)$$

Assuming that the parameters $\underline{\omega}$ are unknown and independent, and then the *ML prediction density* (MLPD) of $T_{(s)}$ given $\underline{\hat{\omega}}_{ML}$ can be obtained using the conditional pdf of the s^{th} order statistic which is given by (18) after replacing the vector of parameters $\underline{\omega}$ by their ML estimators $\underline{\hat{\omega}}_{ML}$, as follows:

$$f_{s:r}(t_{(s)};\underline{\hat{\omega}}_{ML}) = C_{s:r}\hat{\beta} ln(\hat{\alpha})\hat{\alpha}^{1-t_{(s)}^{-\hat{\beta}}}t_{(s)}^{-\hat{\beta}-1}\sum_{j=0}^{r-s}(-1)^j\binom{r-s}{j}\left[\frac{\hat{\alpha}^{1-t_{(s)}^{-\hat{\beta}}}-1}{\hat{\alpha}-1}\right]^{s+j-1},$$

$$\hat{\alpha} \neq 1, t_{(s)} > 1; (\beta > 0), \tag{19}$$

where

$$\underline{\hat{\omega}}_{ML} = (\hat{\alpha}, \hat{\beta}). \tag{20}$$

3.2.1. Point predication

The *ML predictor* (MLP) for the future observation $T_{(s)}$, based on progressive Type-II censored sample can be derived using (19) as follows:

$$\hat{t}_{(s)(ML)} = E\left(t_{(s)}; \underline{\hat{\omega}}_{ML}\right) = \int_{t_{(s)}} t_{(s)} f_{s:r}\left(t_{(s)}; \underline{\hat{\omega}}_{ML}\right) dt_{(s)} =$$

$$\hat{\beta} \ln(\hat{\alpha}) \sum_{j=0}^{r-s} C_{s:r} (-1)^j \binom{r-s}{j} \int_{t_{(s)}} \hat{\alpha}^{1-t_{(s)}^{-\hat{\beta}}} \left(t_{(s)}\right)^{-\hat{\beta}} \left[\frac{\hat{\alpha}^{1-t_{(s)}^{-\hat{\beta}}} - 1}{\hat{\alpha} - 1}\right]^{s+j-1} dt_{(s)} \tag{21}$$

3.2.2 Interval prediction

A $100(1-\tau)\%$ *ML predictive bound* (MLPB) for the future observation $T_{(s)}$, such that $P\left(L_{(s)}(\underline{x}) < T_{(s)} < U_{(s)}(\underline{x}) | \underline{x}\right) = 1 - \tau$, are given by:

$$P\left(T_{(s)} > L_{(s)}(\underline{x}) | \underline{x}\right) = \int_{L_{(s)}(\underline{x})}^{\infty} f_{s:r}\left(t_{(s)}; \underline{\hat{\omega}}_{ML}\right) dt_{(s)} = 1 - \frac{\tau}{2}, \tag{22}$$

and

$$P\left(T_{(s)} > U_{(s)}(\underline{x}) | \underline{x}\right) = \int_{U_{(s)}(\underline{x})}^{\infty} f_{s:r}\left(t_{(s)}; \underline{\hat{\omega}}_{ML}\right) dt_{(s)} = \frac{\tau}{2}. \tag{23}$$

Substituting (19) in (22) and (23), the MLPB are obtained as follows:

$$P(T_{(s)} > L_{(s)}(\underline{x})|\underline{x})$$

$$= \hat{\beta}\ln(\hat{\alpha})\sum_{j=0}^{r-s} C_{s:r}(-1)^j \binom{r-s}{j} \int_{L_{(s)}(\underline{x})}^{\infty} \hat{\alpha}^{1-t_{(s)}^{-\hat{\beta}}} (t_{(s)})^{-\hat{\beta}-1} \left[\frac{\hat{\alpha}^{1-t_{(s)}^{-\hat{\beta}}} - 1}{\hat{\alpha}-1}\right]^{s+j-1} dt_{(s)}$$

$$= 1 - \frac{\tau}{2}, \qquad (24)$$

and

$$P(T_{(s)} > U_{(s)}(\underline{x})|\underline{x}) =$$

$$\hat{\beta}\ln(\hat{\alpha})\sum_{j=0}^{r-s} C_{s:r}(-1)^j \binom{r-s}{j} \int_{U_{(s)}(\underline{x})}^{\infty} \hat{\alpha}^{1-t_{(s)}^{-\hat{\beta}}} (t_{(s)})^{-\hat{\beta}-1} \left[\frac{\hat{\alpha}^{1-t_{(s)}^{-\hat{\beta}}} - 1}{\hat{\alpha}-1}\right]^{s+j-1} dt_{(s)}$$

$$= \frac{\tau}{2}, \qquad (25)$$

Remark:

If $s = 1$ and $s = \frac{r+1}{2}$ in (21), one can predict the minimum observable, $T_{(1)}$, and the median observable when r is odd, $T_{\left(\frac{r+1}{2}\right)}$.

4. Numerical Illustration

This section aims to investigate the precision of the theoretical results of estimation on basis of the simulated and real data.

4.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates on the basis of generated data from the APP distribution. The ML averages of the parameters, rf and hrf based on progressive Type-II censoring scheme are computed. Moreover, CIs of the parameters, rf and hrf are calculated. Simulation studies are performed using Mathcad 15 for illustrating the obtained results.

Applying the algorithm in Balakrishnan and Sandhu (1995), the following steps are used to generate a progressive Type-II censored sample from the APP distribution as follows:

Step 1: Generate m independent $U(0,1)$ random variables $U_1, U_2, \dots, U_m$.

Step 2: For given values of the progressive censoring scheme $R_1, R_2, \dots, R_m$, set
$Y_i = U_i^{1/(i+\sum_{j=m-i+1}^{m} R_j)}$, for $i = 1,2,\dots,m.$

Step 3: Set $UP_i = 1 - (Y_m Y_{m-1} Y_{m-2} \dots Y_{m-i+1}), i = 1,2,\dots,m.$

Then, $UP_1, UP_2, \dots, UP_m$ are progressive Type-II censored sample of size m from $U(0,1)$ distribution.

Step 4: For given values of the parameters α and β, the inverse cdf method, can be used to generate m progressive Type-II censored sample from APP whose cdf is given by (1). Thus, by solving the nonlinear equation

$$x_i = \left(1 - \frac{\ln(UP_i(\alpha-1)+1)}{\ln(\alpha)}\right)^{-\frac{1}{\beta}}, \qquad i = 1,2,\dots,m,$$

the resulting set, $(x_1, x_2, \dots, x_m)$ is the required progressive Type-II censored sample of size m from APP distribution and this obtained sample is ordered.

Step 5: Repeat all the previous steps N times where N represents a fixed number of simulated samples.

a) Evaluating the performance of the estimates is considered through some measurements of accuracy. In order to study the precision and variation of the estimates, it is convenient to use the *estimated risk*

ER $= \frac{\sum_{i=1}^{N}(\text{estimated value} - \text{true value})^2}{N}$, the *relative absolute bias* RAB $= \frac{|\text{estimate-population parameter}|}{\text{population parameter}}$ and *standard error* (SE) $= \sqrt{\text{ER}}$.

b) The ML predictor points and interval for a future observation from the APP distribution based on progressive Type-II censored sample are computed for the two-sample case.

c) Simulation results of ML estimates are displayed in Tables 2-8, where
N = 2000 is the number of repetitions and the samples of size (n=30, 60, 100). For each sample size, $m = 5$, and set of different samples schemes, where

Scheme I: $R_1 = R_2 = \cdots = R_{m-1} = 1$ and $R_m = n - 2m + 1$,

Scheme II: $R_1 = 1, R_2 = 0, R_3 = 1, R_{m-1} = 0$ and $R_m = n - 2m + 1$.

Scheme III: $R_1 = 2, R_2 = 0, R_3 = 1, R_{m-1} = 1$ and $R_m = n - 2m + 1$.

The best scheme is the scheme which minimizes the ERs, SEs and length of CIs of the estimates. Tables 2-4 display the ML averages, ERs, SEs and CIs of the unknown parameters based on progressive Type-II censoring under different samples schemes. While Tables 5-7 present the ML averages, ERs, SEs and CIs of the rf and hrf based on progressive Type-II censoring under different samples schemes. Table 8 gives the ML averages of rf and hrf under different time x_0 at different samples size based on progressive Type-II censoring under different samples schemes.

4.2 Application to real life data

The main aim of this subsection is to demonstrate how the proposed APP distribution based on progressive Type-II censoring can be used in practice. The real lifetime data set is used for this purpose. Ihtisham *et al*. (2019) tested this data using Kolmogorov-Smirnov goodness of fit test through Adequacy Model package of R software, to check the performance of APP distribution. The dataset have been analyzed to demonstrate the performance of the proposed model.

The data set consists of 40 wind related catastrophes used by Hogg and Klugman (1984). It includes claims of $2,000,000. The sorted values, observed in millions are: 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 8, 8, 9, 15, 17, 22, 23, 24, 24, 25, 27, 32, 43.

The K-S test statistic is 0.16 with the p-value = 0.0.2497, the p value showed that the proposed model fits the data very well.

Table 11 displays the ML estimates and SEs of the unknown parameters for the real data set based on progressive Type-II censoring under different samples schemes.

Table 1: ML averages, estimated risks, relative absolute biases, standard errors and 95% confidence intervals of the APP parameters based on progressive Type II censoring under Scheme I (N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$)

N	*Parameters*	*Averages*	ERs	RAB	SE	LL	UL	Length
30	α β	0.508 0.092	0.019 4.782E-3	0.371 2.057	0.138 0.092	0.507 0.076	0.508 0.107	0.001 0.031
60	α β	0.502 0.083	0.017 3.833 E-3	0.356 1.752	0.132 0.062	0.501 0.071	0.502 0.094	0.001 0.023
100	α β	0.499 0.089	0.017 5.330 E-3	0.349 1.961	0.129 0.073	0.499 0.077	0.499 0.100	0.000 0.023

Table 2: ML averages, estimated risks, relative absolute biases, standard errors and 95% confidence intervals of the APP parameters based on progressive Type II censoring under Scheme II (N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$)

N	*Parameters*	*Averages*	ERs	RAB	SE	LL	UL	Length
30	α β	0.509 0.091	0.019 4.603 E-3	0.375 2.043	0.139 0.068	0.508 0.077	0.509 0.106	0.001 0.029
60	α β	0.502 0.084	0.017 3.925 E-3	0.356 1.784	0.132 0.063	0.501 0.072	0.502 0.095	0.001 0.023
100	α β	0.499 0.087	0.017 5.090 E-3	0.349 1.914	0.129 0.071	0.498 0.076	0.500 0.099	0.002 0.023

Table 3: ML averages, estimated risks, relative absolute biases, standard errors and 95% confidence intervals of the APP parameters based on progressive Type II censoring under Scheme III (N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$)

N	*Parameters*	*Averages*	ERs	RAB	SE	LL	UL	Length
30	α β	0.508 0.091	0.019 4.730E-3	0.372 2.0338	0.138 0.069	0.507 0.076	0.508 0.107	0.001 0.031
60	α β	0.501 0.085	0.017 4.184E-3	0.355 1.82	0.131 0.065	0.501 0.072	0.502 0.097	0.001 0.025
100	α β	0.499 0.091	0.017 5.876E-3	0.348 2.037	0.129 0.077	0.498 0.079	0.499 0.104	0.001 0.025

Table 4: ML averages, relative absolute biases, standard errors and 95% confidence intervals of the reliability and hazard rate functions based on progressive Type II censoring under Scheme I (N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$ and $x_0 = 3.5$)

n	rf and hrf	*Averages*	RAB	SE	LL	UL	Length
30	$R(x_0)$ $h(x_0)$	0.857 0.035	0.091 5.148	1.838E-3 1.295E-4	0. 852 0.034	0.862 0.035	0.010 0.001
60	$R(x_0)$ $h(x_0)$	0.870 0.032	0.077 4.762	2.09E-3 1.447E-4	0.864 0.031	0.876 0.032	0.012 0.001
100	$R(x_0)$ $h(x_0)$	0.862 0.034	0.086 5.037	3.455E-3 2.497E-4	0.852 0.033	0.871 0.035	0.019 0.002

Table 5: ML averages, relative absolute biases, standard errors and 95% confidence intervals of the reliability and hazard rate functions based on progressive Type II censoring under Scheme II (N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$ and $x_0 = 3.5, 3$)

N	rf and hrf	*Averages*	RAB	SE	LL	UL	Length
30	$R(x_0)$	0.858	0.090	1.616 E-3	0.853	0. 862	0.009
	$h(x_0)$	0.035	5.127	1.130 E-4	0.034	0.035	0.001
60	$R(x_0)$	0.869	0.079	0.858	0.863	0.874	0.011
	$h(x_0)$	0.037	4.811	0.035	0.037	0.038	0.001
100	$R(x_0)$	0.879	0.075	2.707E-3	0.872	0.886	0.014
	$h(x_0)$	0.039	4.985	3.323E-4	0.038	0.040	0.002

Table 6: ML averages, relative absolute biases, standard errors and 95% confidence intervals of the reliability and hazard rate functions based on progressive Type II censoring under Scheme III (N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$ and $x_0 = 3.5$)

N	rf and hrf	*Averages*	RAB	SE	LL	UL	Length
30	$R(x_0)$	0.87٤	0.٠8٠	١.5٢١ E-3	0.8٧٠	0.8٧8	0.0٠8
	$h(x_0)$	0.0٤١	٥.١٣٢	١.8٢٩ E-4	0.040	0.04١	0.001
60	$R(x_0)$	0.882	0.071	١. 866E-3	0.877	0.887	0.010
	$h(x_0)$	0.038	4.858	2.٢42 E-4	0.037	0.038	0.001
100	$R(x_0)$	0.87٤	0.٠79	3.149 E-3	0.866	0.883	0.017
	$h(x_0)$	0.0٤١	٥.١47	3.937 E-4	0.040	0.042	0.002

Table 7: ML averages of the reliability and hazard rate functions under different time x_0, different samples size based on progressive Type II censoring under different samples schemes (N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$)

n	x_0	Average					
		Scheme I		Scheme II		Scheme III	
		$R(x_0)$	$h(x_0)$	$R(x_0)$	$h(x_0)$	$R(x_0)$	$h(x_0)$
30	3	0.873	0.041	0.874	0.041	0.874	0.041
	3.5	0.857	0.035	0.858	0.035	0.858	0.035
	4	0.844	0.030	0.844	0.030	0.84٥	0.030
	4.5	0.832	0.027	0.833	0.027	0.833	0.027
60	3	0.885	0.037	0.883	0.037	0.882	0.038
	3.5	0.870	0.032	0.869	0.032	0.887	0.032
	4	0.858	0.028	0.856	0.028	0.855	0.028
	4.5	0.847	0.024	0.845	0.025	0.844	0.025
100	3	0.877	0.040	0.879	0.039	0.874	0.041
	3.5	0.862	0.034	0.864	0.033	0.859	0.035
	4	0.849	0.030	0.851	0.029	0.845	0.030
	4.5	0.837	0.026	0.840	0.026	0.834	0.027

Table 8: ML predictive and bounds of the future observation under two-sample prediction based on progressive Type II censoring under Scheme I
(N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$)

n	s	$\hat{t}_{(s)}(ML)$	LL	UL	Length
30	1	0.786	0.716	0.857	0.141
	13	101.219	0000	4041	4041
60	1	0.612	0.606	0.618	0.012
	28	94.952	0000	1364	1364
100	1	0.615	0.606	0.623	0.017
	28	85.386	0000	1424	1424

Table 9: ML predictive and bounds of the future observation under two-sample prediction based on progressive Type II censoring under Scheme II
(N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$)

n	s	$\hat{t}_{(s)}(ML)$	LL	UL	Length
30	1	0.781	0.722	0.840	0.018
	13	103.157	0000	3678	3678
60	1	0.610	0.604	0.615	0.001
	28	97.252	0000	1347	1347
100	1	0.616	0.607	0.625	0.018
	26	92.439	0000	1359	1359

Table 10: ML predictive and bounds of the future observation under two-sample prediction based on progressive Type II censoring under Scheme III
(N=800, $m = 5$, $\alpha = 0.37$ and $\beta = 0.03$)

n	s	$\hat{t}_{(s)}(ML)$	LL	UL	Length
30	١	٠.٧٩١	٠.٧٢١	0.8٦١	٠.١٤٠
	١٣	١٠٠.٦٧١	٠٠٠٠	٤١٣٥	٤١٣٥
60	1	0.609	0.604	0.614	0.010
	28	96.643	0000	1350	1350
100	1	0.613	0.604	0.621	0.017
	28	85. 839	0000	1441	1441

Table 11: ML estimates and standard errors of the parameters for the real data set based on progressive Type II censoring under different samples schemes
(n=38, $m = 5$, α =0.489 and $\beta = 0.198$)

Parameters	**Application**					
	Scheme I		**Scheme II**		**Scheme III**	
	Estimate	**SE**	**Estimate**	**SE**	**Estimate**	**SE**
α	0.489	0.708	0.488	0.740	0.489	0.708
β	0.198	0.042	0.205	0.046	0.198	0.042

Table 12: ML averages of the reliability and hazard rate functions for the real data set under different time x_0, based on progressive Type II censoring under different samples schemes (n=38, $m = 5$, α =0.489 and $\beta = 0.198$)

x_0	Application					
	Scheme I		Scheme II		Scheme III	
	$R(x_0)$	$h(x_0)$	$R(x_0)$	$h(x_0)$	$R(x_0)$	$h(x_0)$
5.0	0.653	0.051	0.643	0.053	0.653	0.051
5.5	0.637	0.046	0.627	0.048	0.637	0.046
6.0	0.623	0.042	0.613	0.043	0.623	0.042
6.5	0.611	0.039	0.601	0.040	0.611	0.039

Table 13: ML predictive of the future observation under two-sample prediction for the real data set based on progressive Type II censoring under different samples schemes (n=37, $m = 5$, α =0.489 and $\beta = 0.198$)

Application					
Scheme I		Scheme II		Scheme III	
s	$\hat{t}_{(s)}(ML)$	s	$\hat{t}_{(s)}(ML)$	s	$\hat{t}_{(s)}(ML)$
1	0.567	1	0.565	1	0.567
19	10.443	19	9.33	19	10.443

5.3 Concluding remarks

- It is noticed, from Tables 2-3, that the ML averages are very close to the initial values of the parameters as the sample size increases. Also, ERs and SEs are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the population parameter values as the sample size increases.

- The lengths of the CIs of the parameters, rf and hrf become narrower as the sample size increases.

- From Table 7 the estimated values of the rf decreases when the time x_0 increases, while the estimated values of the hrf are monotonically decreasing then it gets approximately constant when the time x_0 increases.

- Scheme I is the best censoring scheme where it has the lowest ERs and SEs and the narrower lengths of the CIs.

2. General Conclusion

In this study, therefore the APP distribution is a very flexible reliability model. The ML estimators for the parameters, rf and hrf of the APP distribution based on progressive Type-II censored data are obtained. Monte Carlo simulation is performed. In general, ML estimators of the model parameters, rf and hrf when the sample size *n* increases, the ERs, REs and the lengths of the CIs decrease. Also, when the different values of time x_0 increases, the rf decreases while the hrf monotonically decreases then it approximately stays constant. Point and bounds prediction using the ML prediction method for a future observation based on two-sample prediction are studied. From the results, one can clearly observe that the estimates have smaller ERs. As the sample size increases the ERs and the intervals of the parameters decrease. Moreover, the estimated values of the rf decreases when the time t_0 increases. This indicates that the ML estimates provide asymptotically normally distributed and consistent estimators for the parameters.

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