Bayesian and Non-Bayesian Approach for Mixture of Two Weibull Distributions from Record Values

Amina I. Abo-Hussien

Mervat M. El-Gohary

Abeer A. El-helbawy

Heba H. Mohamed

Department of Statistics, Faculty of Commerce, Al-Azhar University (Girls' Branch)

Abstract

This paper is concerned with estimating the parameters of the finite mixture of two Weibull distributions based on record values. The Maximum likelihood and Bayes methods of estimation are used. Bayes estimates are obtained for the parameters of the mixture model based on beta conjugate prior for the proportion parameter and gamma conjugate priors for the shape parameters under squared-error loss function (as a symmetric loss function) and zero-one loss functions (as asymmetric loss functions) .The Bayes estimates are compared with their corresponding maximum likelihood estimates based on a Monte Carlo simulation study.

Keywords *Weibull distribution; Mixture; Record values; Maximum likelihood and Bayes estimation; Symmetric and asymmetric loss functions; Lindley approximation; Monte Carlo simulation.*

1. Introduction

The study of record values was introduced by Chandler (1952) and he documented many of the basic properties of records. Record values and the associated statistics are of interest and importance in many areas of real life applications involving data relating to industry, economics, biomedical sciences, engineering, the environmental sciences, actuarial sciences, management sciences, social sciences, athletic events, life testing, meteorology, hydrology, seismology and mining. Record values can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of observations (for example the highest rate of the river floating, the highest score of players at many sports, the hottest day, the longest winning streak in professional basketball, the lowest stock market figure, …etc.). Records and associated statistics have been studied by Ahsanullah (1993, 1995) and Arnold *et al.* (1992,1998).We encounter this notion frequently in daily life, especially in singling out record values from a set of others and in registering and recalling record values.

Mixtures of lifetime distributions occur when two different causes of failure are present, each with the same parametric form of lifetime distributions. The finite mixtures of lifetime distributions have proved to be of considerable interest both in terms of their methodological development and practical applications, see McLachlan and Basford (1988), McLachlan and Peel (2000) and Abu-Zinadah (2010). Inferences on finite mixture models when the components belong to the same family were studied by Ahmad (1982), Amoh (1983), Al-Hussaini and Ahmad (1984), Al-Hussaini (1999), Al-Hussaini *et al.* (2000) and Jaheen (2005), among others.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Set

$Y_m = \max(X_1, X_2, \ldots, X_m)$, $m \geq 1$ .We denote X_j is an upper record value of the sequence $\{X_m, m \geq 1\}$, if $Y_j > Y_{j-1}$, $j > 1$. Upper record values can be transformed to lower record values by replacing the original sequence of random variables $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or if $P(X_j > 0) = 1$ by $\{1/X_i, i \geq 1\}$. The notation $X_{U(m)}$ is used for the *m* th upper record value.

A random variable X is said to follow a finite mixture distribution with k components, if the probability density function of X can be written in the form:

$$f(x) = \sum_{j=1}^{k} p_j f_j(x) \tag{1}$$

where p_j is a non-negative real number (known as the *j* th mixing proportion) such that $\sum_{j=1}^{k} p_j = 1$ and f_j is a density function (known as the *j* th component), $j = 1,2,\ldots,k$.

The probability density function (pdf) and the cumulative distribution function (cdf) of a mixture of two Weibull distributions are given by

$$f(x;p,\lambda_1,\lambda_2) = pf_1(x;\lambda_1) + (1-p)f_2(x;\lambda_2), \quad (2)$$

$$F(x;p,\lambda_1,\lambda_2) = pF_1(x;\lambda_1) + (1-p)F_2(x;\lambda_2), \quad (3)$$

and

$$R(x;p,\lambda_1,\lambda_2) = 1 - F(x;p,\lambda_1,\lambda_2). \quad (4)$$

where for $j=1,2$, $f_j(x;\lambda_j)$, $F_j(x;\lambda_j)$ and $R_j(x;\lambda_j)$ are the pdf , cdf and the reliability function of the Weibull distribution which are given respectively by

$$f_j(x;p,\lambda_1,\lambda_2) = \lambda_j\, x^{\lambda_j-1} e^{-x^{\lambda_j}}, \quad x>0, \lambda_j>0, \quad (5)$$

$$F_j(x;p,\lambda_1,\lambda_2) = 1 - e^{-x^{\lambda_j}}, \quad (6)$$

$$R_j(x;\lambda_1,\lambda_2) = 1 - F(x;p,\lambda_1,\lambda_2) = e^{-x^{\lambda_j}}. \quad (7)$$

We shall denote the vector of parameters of the mixture model $\psi = (p,\lambda_1,\lambda_2)$.

The maximum likelihood estimates are obtained in Section 2. In Section 3 the approximate form of Lindley is used for obtaining the Bayes estimates for the vector of parameters $\psi = (p,\lambda_1,\lambda_2)$ of the finite mixture of two Weibull distributions based on upper record values under squared-error and zero-one loss functions. Bayes estimates are compared with their corresponding maximum likelihood estimates based on a Monte Carlo simulation study in Section 4. Concluding remarks are given in Section 5.

2. Maximum Likelihood Estimation

Observing m upper record values $X_{U(1)} = x_1, X_{U(2)} = x_2, \dots, X_{U(m)} = x_m$ from a finite mixture of two Weibull distributions with the pdf given in equation (2). The likelihood function (LF) is given by

$$l(\psi;\underline{x}) = f(x_m;\psi) \prod_{i=1}^{m-1} \frac{f(x_i;\psi)}{R(x_i;\psi)}, \quad (8)$$

where $\underline{x} = (x_1, x_2, \dots, x_m)$.

The natural logarithm of the LF may be written as

$L(\psi; \underline{x}) = \ln l(\psi; \underline{x}) = \sum_{i=1}^{m} \ln f(x_i;\psi) - \sum_{i=1}^{m-1} \ln R(x_i;\psi)$. (9)

Differentiating $L(\psi; \underline{x})$ with respect to $p, \lambda_1, and \ \lambda_2$, respectively, and then equating to zero, one can obtain the following likelihood equations:

$$\frac{\partial L}{\partial p} = \sum_{i=1}^{m} A(x_i) - \sum_{i=1}^{m-1} C(x_i) = 0 \ , \quad (10)$$

$$\frac{\partial L}{\partial \lambda_1} = p[\sum_{i=1}^{m} B_1(x_i)a_1(x_i) + \sum_{i=1}^{m-1} D_1(x_i)b_1(x_i)] = 0 \ , \quad (11)$$

$$\frac{\partial L}{\partial \lambda_2} = (1-p)[\sum_{i=1}^{m} B_2(x_i)a_2(x_i) + \sum_{i=1}^{m-1} D_2(x_i)b_2(x_i)] = 0 \ , \quad (12)$$

where for $j = 1, 2$ and $i = 1, 2, \dots, m$.

$$A(x_i) = B_1(x_i) - B_2(x_i) \ , \qquad C(x_i) = D_1(x_i) - D_2(x_i) \ . \quad (13)$$

$$B_j(x_i) = \frac{f_j(x_i)}{f(x_i,\psi)} \ , \qquad D_j(x_i) = \frac{R_j(x_i)}{R(x_i,\psi)} \quad (14)$$

$$a_j(x_i) = \frac{1+\lambda_1 \ln x_i - \lambda_1 (x_i)^{\lambda_1} \ln x_i}{\lambda_j} \ , \quad b_j(x_i) = (x_i)^{\lambda_1} \ln x_i \ . \quad (15)$$

To obtain the maximum likelihood estimate of the vector of parameters

$\psi = (p, \lambda_1, \lambda_2)$, the system of nonlinear equations (10 -12) can be solved numerically.

3. Bayes Estimation

Assuming that the parameters $p, \lambda_1 \ and \ \lambda_2$ are independent random variables, then the joint prior density of the random vector $\psi = (p, \lambda_1, \lambda_2)$ is given by

$$g_j(p, \lambda_1, \lambda_2) = g_1(\lambda_1)\, g_2(\lambda_2)\, g_3(p) \quad (16)$$

where for j=1, 2 , $g_j(\lambda_j)$ is a prior density of λ_j and $g_3(p)$ is a prior density of p .

Let $p\sim$ Beta (a, b) and the gamma conjugate prior density with parameters α_j *and* β_j is chosen as a prior for λ_j , j = 1, 2, with the following pdf

$$g_j(\lambda_j) = \frac{\beta_j}{(\alpha_j-1)!}\ \lambda_j^{\alpha_j-1}\, e^{-\beta_j\lambda_j} \ , \qquad \lambda_j > 0 \qquad , (17)$$

where for j = 1, 2 , $\alpha_j > 0$ *and* $\beta_j > 0$,

and

$$g_3(p) = \frac{1}{B(\alpha,\beta)}\, p^{a-1}\,(1-p)^{b-1} \ , \qquad 0 < p < 1 , \quad (18)$$

where *a and b* are real numbers.

Hence the joint prior density of $p,\ \lambda_1$ *and* λ_2 is given by

$$g(p,\ \lambda_1\ ,\lambda_2) \propto\ p^{a-1}\,(1-p)^{b-1}\ \lambda_1^{\alpha_1-1}\,\lambda_2^{\alpha_2-1}\ e^{-(\beta_1\lambda_1+\beta_2\lambda_2)} \ ,$$

$$0<p<1,\ \lambda_1>0,\ \lambda_2>0 \qquad (19)$$

where a, b, and α_j and β_j , j = 1,2 are real numbers.

The joint prior density in equation (19) has been used by Al-Hussaini (1999) and is chosen such that it would be rich enough to cover the prior belief of the experimenter.

The posterior density function of ψ given $\underline{x}$ is obtained by combining equations (8) and (19), and can be written as

$$q(\psi;\underline{x}) \propto p^{a-1}\,(1-p)^{b-1}\ \lambda_1^{\alpha_1-1}\,\lambda_2^{\alpha_2-1}\ e^{-(\beta_1\lambda_1+\beta_2\lambda_2)}$$
$$f(x_m;\ \psi)\ \prod_{i=1}^{m-1} \frac{f(x_i;\psi)}{R(x_i;\psi)} \ . \qquad (20)$$

3.1 Bayesian estimation under squared error loss function

Considering squared error loss function, the Bayesian estimator is the posterior mean of the corresponding posterior density function and is given by a ratio of two integrals. Both integrals can't be obtained in a simple closed form so numerical integration technique must be used, which can be computationally intensive, especially in high dimensional parameter space. Hence Lindley's approximation (1980) is used to obtain the Bayes estimators.

Lindley (1980) showed that the approximate Bayes estimate of ψ about the posterior mode $\tilde{\psi}$ is of the form

$$\hat{u}_B(\psi) \equiv E(u(\psi)|x) \approx u(\tilde{\psi}) + \frac{1}{2}\sum_{i,j=1}^{N} u_{ij}(\tilde{\psi})\tau_{ij} + \frac{1}{2}\sum_{i,j,k,l=1}^{N} \Delta_{ijk}(\tilde{\psi})\tau_{ij}\tau_{kl}\, u_l(\tilde{\psi}), \quad (21)$$

where for *i, j, k =1,2,…,N*,

$$u_i(\tilde{\psi}) = \frac{\partial u(\psi)}{\partial \psi_i}, \quad u_{ij}(\tilde{\psi}) = \frac{\partial^2 u(\psi)}{\partial \psi_i \partial \psi_j}, \quad (22)$$

$$\Delta_{ij}(\tilde{\psi}) = \frac{\partial^2 \Delta(\psi)}{\partial \psi_i \partial \psi_j}, \quad \Delta_{ijk}(\tilde{\psi}) = \frac{\partial^3 \Delta(\psi)}{\partial \psi_i \partial \psi_j \partial \psi_k}, \quad (23)$$

$$\Sigma_{N\times N} = (\tau_{ij}) = (-\Delta_{ij})^{-1}_{N\times N} \quad (24)$$

where $\Delta(\psi|x)$ is the logarithm of the posterior density .

All functions in equations (21-24) are evaluated at $\tilde{\psi}$, the mode of the posterior density.

This approximation form of Lindley has been used by many authors for obtaining Bayes estimators for the parameters of some lifetime distributions.

For three parameters Lindley's approximation in (21) reduces to the following

$$\hat{u}_B(\psi) \approx u(\tilde{\psi}) + \frac{1}{2}\sum_{i,j=1}^{3} u_{ij}(\tilde{\psi})\tau_{ij} + \frac{1}{2}\sum_{i,j,k,l=1}^{3} \Delta_{ijk}(\tilde{\psi})\tau_{ij}\tau_{kl}\, u_l(\tilde{\psi}) \quad (25)$$

The logarithm of the posterior density function in (20) is given by

$$\Delta(\psi|x) \propto (a-1)\ln p + (b-1)\ln(1-p) - (\beta_1\lambda_1 + \beta_2\lambda_2)$$
$$+ \sum_{j=1}^{2}(\alpha_j - 1)\ln\lambda_j + \sum_{i=1}^{m} \ln f(x_i;\psi) - \sum_{i=1}^{m-1} \ln R(x_i;\psi) \quad . (26)$$

The following nonlinear equations can be solved to obtain the mode $\tilde{\psi} = (\tilde{p}, \tilde{\lambda}_1, \tilde{\lambda}_2)$ of the posterior density

$$\frac{\partial\Delta}{\partial p} = \frac{a-1}{p} - \frac{b-1}{1-p} + \frac{\partial L}{\partial p} = 0 \ , \qquad (27)$$

$$\frac{\partial\Delta}{\partial\lambda_1} = -\beta_1 + \frac{\alpha_1 - 1}{\lambda_1} + \frac{\partial L}{\partial\lambda_1} = 0 \ , \qquad (28)$$

$$\frac{\partial\Delta}{\partial\lambda_2} = -\beta_2 + \frac{\alpha_2 - 1}{\lambda_2} + \frac{\partial L}{\partial\lambda_2} = 0 \ , \qquad (29)$$

where $\frac{\partial L}{\partial p}$, $\frac{\partial L}{\partial\lambda_1}$ and $\frac{\partial L}{\partial\lambda_2}$ are given in equations (10-12) .

A solution of the system of equations (27-29) can be obtained iteratively by choosing initial values $p^{(0)}, \lambda_1{}^{(0)}$ and $\lambda_2{}^{(0)}$ of p, λ_1 and λ_2 respectively.

To apply Lindley's approximation form (25), we have to obtain the elements of the matrix $(\Delta_{ij})_{3\times3}$, where for *i, j*=1, 2, 3, Δ_{ij} are given in the Appendix. The elements τ_{ij}, *i, j*=1,2,3, of the matrix Σ are obtained numerically by inverting the matrix $(-\Delta_{ij})$. Furthermore, for *i, j, k*=1,2,3, the elements Δ_{ijk} can be obtained from equations (27-29) and are given in the Appendix.

If we set $u(\psi) = p, \lambda_1$ and λ_2 in equation (25), we can write the approximate Bayes estimators for p, λ_1 and λ_2 as follows:

$$\hat{p}_{Bs} = \tilde{p} + \frac{1}{2}\sum_{i,j,k,l=1}^{3} \Delta_{ijk}(\tilde{\psi})\tau_{ij}\tau_{k1} \ , \qquad (30)$$

$$\hat{\lambda}_{1Bs} = \tilde{\lambda}_1 + \frac{1}{2}\sum_{i,j,k,l=1}^{3} \Delta_{ijk}(\tilde{\psi})\tau_{ij}\tau_{k2} \ , \qquad (31)$$

$$\hat{\lambda}_{2Bs} = \tilde{\lambda}_2 + \frac{1}{2}\sum_{i,j,k,l=1}^{3} \Delta_{ijk}(\tilde{\psi})\tau_{ij}\tau_{k3} \ . \qquad (32)$$

3.2 Bayesian estimation under zero-one loss function

Considering the zero-one loss function, then the Bayesian estimator is the mode of the corresponding posterior density function.

Taking logarithms of both sides of equation (20) yields equation (26) which we have obtained in Subsection 3.1. Differentiating $\Delta(\psi|x)$ with respect to p, λ_1 *and* λ_2 and equating to zero we can obtain the nonlinear equations (27-29), then solving numerically, the mode can be obtained.

4. Monte Carlo Simulation

The Bayes and maximum likelihood estimates of the vector of parameters $\psi = (p, \lambda_1, \lambda_2)$ are computed and compared according to the following steps:

1. The population parameters p, λ_1 *and* λ_2 are generated from the joint prior density given by equation (19), for given values of the prior parameters $a, b, \alpha_1, \beta_1, \alpha_2,$ *and* β_2. The MATHCAD 7 Program is used in the generation.
2. Based on the generated values of p, λ_1 *and* λ_2, upper record values of size *m*= 3, 4, 5, 6 and 7 are generated from the finite mixture of two Weibull distributions with the pdf given in equation (2).
3. The maximum likelihood estimates (MLEs) of p, λ_1 *and* λ_2 are computed by solving the nonlinear equations (10-12).The MATHCAD 7 Program is used.
4. For given values of the prior parameters $a, b, \alpha_1, \beta_1, \alpha_2,$ *and* β_2 the Bayes estimates of p, λ_1 *and* λ_2 are computed using equations (27-29).
5. The mean squared errors are computed for different sizes *m*.
6. The above steps are repeated 1000 times and the estimated risks (ER) are computed as follows:

$$\text{ER}(\hat{\psi}) = \sum \frac{(\hat{\psi} - \psi)^2}{10^3} \quad (33)$$

The results are displayed in Tables (4.1), (4.2) and (4.3).

Table (4.1) The variances and ER's of ML and Bayes estimates of p, λ_1 and λ_2

when $p = 0.3,\ \lambda_1 = 2$ and $\lambda_2 = 3$

n	ER of $\hat{p}, \hat{\lambda}_1$ and MLE	ER of B_Z $\hat{p}, \hat{\lambda}_1$ and	ER of B_S $\hat{p}, \hat{\lambda}_1$ and	Var of $\hat{p}, \hat{\lambda}_1$ and MLE	Var of B_Z $\hat{p}, \hat{\lambda}_1$ and	Var of B_S $\hat{p}, \hat{\lambda}_1$ and
3	0.013 0.093 0.054	0.027 0.070 0.476	0.158 0.115 0.215	0.013 0.064 0.043	3.550×10^{-3} 0.063 0.447	7.170×10^{-3} 0.115 0.195
4	0.018 0.167 0.056	0.018 0.062 0.236	0.173 0.253 0.211	0.018 0.166 0.056	4.305×10^{-3} 0.049 0.235	0.010 0.231 0.210
5	0.021 0.277 0.066	9.275×10^{-3} 0.051 0.080	0.185 0.346 0.286	0.018 0.242 0.065	4.626×10^{-3} 0.041 0.078	0.017 0.318 0.281
6	0.023 0.236 0.081	5.884×10^{-3} 0.050 0.042	0.060 0.216 0.514	0.013 0.238 0.078	3.971×10^{-3} 0.045 0.038	0.019 0.136 0.394
7	0.027 0.273 0.115	0.015 0.087 0.224	0.054 0.194 0.508	8.481×10^{-3} 0.195 0.111	0.012 0.085 0.219	0.019 0.141 0.371

Table (4.2) The variances and ER's of ML and Bayes estimates of p, λ_1 and λ_2

when $p = 0.3, \ \lambda_1 = 4 \ and \ \lambda_2 = 5$

n	ER of $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$ MLE	ER of B_Z $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$	ER of B_S $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$	Var of $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$ MLE	Var of B_Z $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$	Var of B_S $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$
3	9.478×10^{-4} 2.503×10^{-3} 1.03×10^{-3}	0.015 0.067 0.037	0.039 0.174 0.234	9.478×10^{-4} 2.493×10^{-3} 9.321×10^{-4}	6.917×10^{-5} 5.687×10^{-3} 0.013	2.682×10^{-3} 0.078 0.233
4	1.295×10^{-3} 0.017 2.829×10^{-3}	0.015 0.081 0.076	0.039 0.182 0.279	1.289×10^{-3} 0.017 2.277×10^{-3}	1.064×10^{-4} 0.025 0.056	1.439×10^{-3} 0.100 0.228
5	5.209×10^{-3} 0.120 0.011	0.016 0.153 0.302	0.046 0.232 0.319	4.657×10^{-3} 0.111 8.772×10^{-3}	5.298×10^{-4} 0.122 0.300	2.376×10^{-3} 0.143 0.227
6	0.014 0.439 0.066	0.022 0.368 1.116	0.059 0.217 0.435	9.31×10^{-3} 0.332 0.049	1.743×10^{-3} 0.364 0.982	4.116×10^{-3} 0.187 0.339
7	0.024 0.687 0.150	0.023 0.546 1.656	0.068 0.215 0.449	9.77×10^{-3} 0.422 0.095	1.850×10^{-3} 0.473 1.191	6.210×10^{-3} 0.215 0.388

Table (4.3) The variances and ER's of ML and Bayes estimates of p, λ_1 and λ_2

when $p = 0.5, \lambda_1 = 2$ and $\lambda_2 = 3$

n	ER of $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$ MLE	ER of B_Z $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$	ER of B_S $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$	Var of $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$ MLE	Var of B_Z $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$	Var of B_S $\hat{p}, \hat{\lambda}_1$ and $\hat{\lambda}_2$
3	0.017 0.080 0.047	5.702×10^{-3} 0.032 0.038	0.066 0.209 0.372	0.015 0.046 0.040	3.425 0.030 0.037	0.011 0.131 0.365
4	0.027 0.118 0.055	4.48×10^{-3} 0.074 0.057	0.069 0.215 0.397	0.018 0.115 0.052	3.047×10^{-3} 0.061 0.057	0.014 0.126 0.360
5	0.038 0.170 0.063	2.921×10^{-3} 0.097 0.102	0.070 0.247 0.471	0.015 0.162 0.061	2.286×10^{-3} 0.082 0.102	0.018 0.145 0.375
6	0.047 0.165 0.065	1.085×10^{-3} 0.050 0.063	0.060 0.216 0.514	9.541×10^{-3} 0.139 0.065	9.878×10^{-4} 0.047 0.062	0.019 0.136 0.394
7	0.053 0.128 0.061	1.484×10^{-3} 0.034 0.088	0.054 0.194 0.508	5.333×10^{-3} 0.105 0.060	1.403×10^{-3} 0.032 0.088	0.019 0.141 0.371

5. Concluding Remarks

1. Estimation of the parameters of the finite mixture model of two Weibull distributions are considered from a Bayesian approach based on record statistics. The Bayes estimates are obtained by using the approximation form of Lindley. These estimates are compared with their corresponding maximum likelihood estimates.

2. It may be noted that the estimated risks of the estimates (ML or Bayes) decrease as the sample size *m* increases.
3. In most cases MLEs tend to be more efficient than the Bayes estimates since they have smaller estimated risks.
4. The MLEs perform better than the Bayes estimates under squared error and zero-one loss functions.
5. The Bayes estimates under zero-one loss function perform better than the corresponding results under squared error loss function.
6. Different values of the prior parameters $a, b, \alpha_1, \beta_1, \alpha_2$, and β_2 (more than those in the tables) have been considered but the previous conclusion did not change.

References

1. Abu-Zinadah, H.h. (2010). A Study on Mixture of Exponentiated Pareto and Exponential Distributions. Journal of Applied Sciences Research, 6(4), 358-376.
2. Ahmad, K.E. (1982). Mixture of Inverse Gaussian distributions. Ph.D. thesis, University of Assiut, Assiut, Egypt.
3. Ahsanullah, M. (1993). On the record values from univariate distributions. National Institute of Standards and Technology, J. Res. Special Publications, 866, 1-6.
4. Ahsanullah, M. (1995). Record Statistics. NOVA Science Publishers Inc., Huntington, New York.
5. Al-Hussaini, E.K. (1999). Bayesian prediction under a mixture of two exponential components model based on type I censoring. Journal of Applied Statistical Science, 8, 173-185.
6. Al-Hussaini, E.K., Al-Dayian, G.R. and Adham, S.A. (2000). On finite mixture of two component Gompertz lifetime model. Journal of Statistical Computation and Simulation, 67, 1-20.
7. Al-Hussaini, E.K. and Ahmad, K.E. (1984). Information matrix for a mixture of two Inverse Gaussian distributions. Journal of Statistical Computation and Simulation, 13, 785-800.
8. Amoh, R.K. (1983). Classification procedures associated with the Inverse Gaussian distribution. Ph.D. thesis, University of Manitoba, Winnipeg, Canada.
9. Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1998). Records. New York: John Wiley and Sons, Inc.
10. Arnold, B.C., Balakrishnan, N. and Nagaraja, wH.N. (1992). A First Course In order Statistics .New York: John Wiley and Sons.Inc.
11. Chandler, K.N. (1952). The distribution and frequency of record values. Journal of the Royal Statistical Society Series B, 14, 220-228.

12. Jaheen, Z.F. (2005). On record statistics from a mixture of two exponential distributions. Journal of Statistical Computation and Simulation, 75(1), 1-11.
13. McLachlan, G.J. and Basford, K.E. (1988). Mixture Models. Wiley, New York.
14. McLachlan, G.J. and Peel, D. (2000). Finite Mixture Models. Wiley, New York.
15. Lindely, D. V. (1980). Approximate Bayesian method. TrabaJos de Estadistica, 31: 223-237.
16. Soliman, A.A., Abd Ellah, A.H. and Sultan, K.S.(2006). Comparison of estimates using record statistics from Weibull model: Bayesian and non-Bayesian approaches. Computational Staistics and Data Analysis, 1-13.

Appendix

It follows from equations (20-22) and equations (25-27) that for *i, j, k = 1, 2, 3,* $\Delta_{ijk} = \Delta_{ikj} = \Delta_{jik} = \Delta_{jki} = \Delta_{kij} = \Delta_{kji}$, which can be shown to be

$$\Delta_{11} = \frac{\partial^2 \Delta}{\partial p^2} = -\frac{a-1}{p^2} - \frac{b-1}{(1-p)^2} - \sum_{i=1}^{n} A^2(x_i) + \sum_{i=1}^{n-1} C^2(x_i) \tag{A1}$$

$$\Delta_{12} = \frac{\partial^2 \Delta}{\partial p \partial c_1} = \sum_{i=1}^{n} B_1(x_i) a_1(x_i)[1 - p(A(x_i)] + \sum_{i=1}^{n-1} x_i^{c_1} \ln x_i D_1(x_i)[1 - p(C(x_i))] = \Delta_{21} \tag{A2}$$

$$\Delta_{13} = \frac{\partial^2 \Delta}{\partial p \partial c_2} = -\sum_{i=1}^{n} B_2(x_i) a_2(x_i)[1 + qA(x_i)] - \sum_{i=1}^{n-1} x_i^{c_2} \ln x_i D_2(x_i)[1 + qC(x_i)] = \Delta_{31} \tag{A3}$$

$$\Delta_{22} = \frac{\partial^2 \Delta}{\partial c_1^2} = \frac{-(\alpha_1 - 1)}{c1^2} + p\sum_{i=1}^{n} B_1(x_i) a_{11}^2(x_i) + p\sum_{i=1}^{n-1} x_i^{c_1} (\ln x_i)^2 D_1(x_i) D_{11}(x_i) \tag{A4}$$

$$\Delta_{23} = \frac{\partial^2 \Delta}{\partial c_1 \partial c_2} = -pq\sum_{i=1}^{n} B_1(x_i) B_2(x_i) a_1(x_i) a_2(x_i) + pq\sum_{i=1}^{n-1} x_i^{c_1 + c_2} (\ln x_i)^2 D_1(x_i) D_2(x_i) = \Delta_{32} \tag{A5}$$

$$\Delta_{33} = \frac{\partial^2 \Delta}{\partial c_2^2} = \frac{-(\alpha_2 - 1)}{c2^2} + q\sum_{i=1}^{n} B_2(x_i) a_{22}(x_i) + q\sum_{i=1}^{n-1} x_i^{c_2} (\ln x_i)^2 D_2(x_i) D_{22}(x_i) \tag{A6}$$

Furthermore, for *i, j, k* = 1, 2, 3 the elements (Δ_{ijk}) are obtained as

$$\Delta_{111} = \frac{2(a-1)}{p^3} - \frac{2(b-1)}{(1-p)^3} + 2\left[\sum_{i=1}^{n} A^3(x_i) - \sum_{i=1}^{n-1} C^3(x_i)\right] \tag{A7}$$

$$\Delta_{112} = 2\sum_{i=1}^{n} B_1(x_i) a_1(x_i) A(x_i)(pA(x_i)-1) + 2\sum_{i=1}^{n-1} (\ln x_i) x_i^{c_1} D_1(x_i) C(x_i)(pC(x_i)-1) \qquad \text{(A 8)}$$

$$\Delta_{113} = 2\sum_{i=1}^{n} B_2(x_i) a_2(x_i) A(x_i)(qA(x_i)+1) + 2\sum_{i=1}^{n-1} (\ln x_i) x_i^{c2} D_2(x_i) C(x_i)(qC(x_i)+1) \qquad \text{(A 9)}$$

$$\Delta_{122} = \sum_{i=1}^{n} B_1(x_i)(1-pA(x_i))(a_{11}(x_i) - pB_1(x_i) a_1^2(x_i)) + \sum_{i=1}^{n-1} (\ln x_i)^2 x_i^{c_1} D_1(x_i)(1-pC(x_i))(D_{11}(x_i) + pD_1(x_i) x_i^{c_1}) \qquad \text{(A 10)}$$

$$\begin{aligned}\Delta_{123} &= \sum_{i=1}^{n} B_1(x_i) B_2(x_i) a_1(x_i) a_2(x_i)[q(pA(x_i)-1) + p(qA(x_i)+1)] \\ &\quad + \sum_{i=1}^{n-1} x_i^{c_1+c_2} (\ln x_i)^2 D_1(x_i) D_2(x_i)[q(1-pC(x_i)) - p(1+qC(x_i))]\end{aligned} \qquad \text{(A 11)}$$

$$\Delta_{133} = -\sum_{i=1}^{n} B_2(x_i)(1+qA(x_i))(a_{22}(x_i) - qB_2(x_i) a_2^2(x_i)) - \sum_{i=1}^{n-1} (\ln x_i)^2 x_i^{c_2} D_2(x_i)(1+qC(x_i))(D_{22}(x_i) + qD_2(x_i) x_i^{c_2}) \qquad \text{(A 12)}$$

$$\begin{aligned}\Delta_{222} = \frac{\partial^3 \Delta}{\partial c_1^3} &= \frac{2(\alpha_1 - 1)}{c1^3} + p\sum_{i=1}^{n} B_1(x_i) \begin{bmatrix} a_{11}(x_i) a_1(x_i)(1-pB_1(x_i)) + 2a_1(x_i) b_1(x_i) \\ + b_{11}(x_i) - pa_1(x_i) B_1(x_i)\big(b_1(x_i) + a_{11}(x_i)\big) \end{bmatrix} \\ &\quad + p\sum_{i=1}^{n-1} x_i^{c_1} (\ln x_i)^3 D_1(x_i)\left[D_{11}^2(x_i) + x_i^{c_1}\left(pD_1(x_i) D_{11}(x_i) - 1\right)\right]\end{aligned} \qquad \text{(A 13)}$$

$$\Delta_{223} = pq\sum_{i=1}^{n} B_1(x_i) B_2(x_i) a_2(x_i)\left[-a_{11}(x_i) + pa_1^2(x_i) B_1(x_i)\right] + pq\sum_{i=1}^{n-1} x_i^{c_1+c_2} (\ln x_i)^3 D_1(x_i) D_2(x_i)\left[px_i^{c_1} D_1(x_i) + D_{11}(x_i)\right] \qquad \text{(A 14)}$$

$$\Delta_{233} = -pq\sum_{i=1}^{n} B_1(x_i)B_2(x_i)a_1(x_i)(a_{22}(x_i) - qa_2^2(x_i)B_2(x_i)) + pq\sum_{i=1}^{n-1} x_i^{c1+c2}(\ln x_i)^3 D_1(x_i)D_2(x_i)\left[D_{22}(x_i) + qx_i^{c_2}D_2(x_i)\right]$$

(A 15)

$$\Delta_{333} = \frac{\partial^3\Delta}{\partial c_2^3} = \frac{2(\alpha_2 - 1)}{c2^3} + q\sum_{i=1}^{n} B_2(x_i)\begin{bmatrix} a_{22}^2(x_i)a_2(x_i)(1 - qB_2(x_i)) + 2a_2(x_i)b_2(x_i) \\ + b_{22}(x_i) - qa_2(x_i)B_2(x_i)\left(b_2(x_i) + a_{22}(x_i)\right) \end{bmatrix}$$
$$+ q\sum_{i=1}^{n-1} x_i^{c_2}(\ln x_i)^3 D_2(x_i)\left[D_{22}^2(x_i) + x_i^{c_2}\left(qD_2(x_i)D_{22}(x_i) - 1\right)\right]$$

(A 16)

where

$a_j(x) = \dfrac{1}{cj} + \ln x - x^{cj}\ln x$, $\quad j = 1,2$,

$b_j(x) = \dfrac{1}{cj^2} - x^{cj}(\ln x)^2$, $\quad j = 1,2$,

$a_{11}(x) = a_1^2(x) + b_1(x) - pa_1^2(x)B_1(x)$, $\quad j = 1,2$,

$a_{22}(x) = a_2^2(x) + b_2(x) - qa_2^2(x)B_2(x)$, $\quad j = 1,2$,

$D_{11}(x) = 1 - x^{c1} + px^{c1}D_1(x)$,

$D_{22}(x) = 1 - x^{c2} + qx^{c2}D_2(x)$.