COMPARATIVE ESTIMATION METHODS FOR EXPONENTIATED WEIBULL DISTRIBUTION

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Abstract

The exponentiated Weibull (EW) distribution was introduced by Mudholkar and Srivastava (1993). This distribution is involving additional parameter to the Weibull distribution. In this paper, we mainly estimate the three parameters of the exponentiated Weibull distribution using different methods of estimation such as: maximum likelihood method, method of moments, percentile estimation and least squares method. A comparison is performed based on variances, relative absolute biases, relative absolute estimate of risks and total deviation. The moment estimators are found to be appropriate than other methods. In addition, the sampling distributions of the moment estimators are obtained. A simulation study and concluding remarks are introduced.

Key Words: *Exponentiated Weibull distribution, Maximum likelihood estimators, Moment estimators, Percentile estimators, Least squares estimators, Total deviation, Sampling distribution.*

1 Introduction

One of the most interesting distributions, called the exponentiated Weibull (EW) distribution was introduced by Mudholkar and Srivastava (1993). This distribution is involving one additional parameter to the Weibull distribution. It has different types of parameters (scale, shape and location). Extensions of the Weibull family were introduced by Weibull in 1939 [see Lawless (2003)]. The EW distribution is an important life testing model; it is more flexible because it has many types of failure rate behaviors. The failure rate function of the EW distribution may be decreasing, constant, increasing or bathtub.

The EW distribution can be used for modeling lifetime data from reliability, various extreme value data, survival and population studies. The applications of the exponentiated Weibull distribution in reliability and survival studies were illustrated by Mudholkar et al. (1995). Its properties have been studied in more detail by Mudholkar and Hutson (1996) and Nassar and Eissa (2003). These studies presented useful applications of the distribution in the modeling of flood data and in reliability.

Assume that the random variable T follows an EW distribution. The cumulative distribution function (cdf), is given by

$$F(t;\theta,\alpha,\lambda) = [1 - e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta} \quad ; \quad t > 0, \theta, \alpha, \lambda > 0 \,, \quad (1)$$

where $(\theta,\alpha) > 0$ denote the shape parameters and $\lambda > 0$ is a scale parameter.

The rest of the paper is organized as follows: In Section (2), some properties of the EW distribution are discussed. In Section (3), the different methods of estimation such as maximum likelihood estimation, moment estimation, percentile estimation and least squares estimation are obtained. Comparisons among estimators are investigated through Monte Carlo simulations in Section (4). In addition, the sampling distributions of the moments estimators are discussed in Section (5). The last section, Section (6) includes some conclusion remarks.

2 Some Statistical Properties of the Exponentiaed Weibull Distribution

If T is a random variable follows an EW distribution. The probability density function (pdf) is given by

$$f(t;\theta,\alpha,\lambda) = \frac{\theta\alpha}{\lambda}[1 - e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta-1} e^{-(\frac{t}{\lambda})^{\alpha}} (\frac{t}{\lambda})^{\alpha-1} \quad ; \quad t, \theta, \alpha, \lambda > 0 \quad (2)$$

The EW distribution is unimodal for fixed α and λ, and it becomes more and more symmetric as θ increases. There are several well known distributions which can be obtained as a special case from EW distribution such as: Weibull $(\theta = 1)$, Exponential distribution $(\theta = 1, \alpha = 1)$, Exponentiated exponential distribution $(\alpha = 1)$, Rayleigh distribution $(\theta = 1, \alpha = 2)$ and Burr type X distribution $(\alpha = 2)$.

1- Reliability Function

The reliability function is given by

$$R(t;\theta,\alpha,\lambda) = 1 - [1 - e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta} \quad ; \quad t,\theta,\alpha,\lambda > 0 \qquad (3)$$

2- Failure Rate Function

The failure rate (FR) function is given by

$$Z(t;\theta,\alpha,\lambda) = \frac{\theta\alpha\,[1-e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta-1}\, e^{-(\frac{t}{\lambda})^{\alpha}}\,(\frac{t}{\lambda})^{\alpha-1}}{\lambda\,[1-[1-e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta}]} \quad ; \quad t,\theta,\alpha,\lambda > 0 \qquad (4)$$

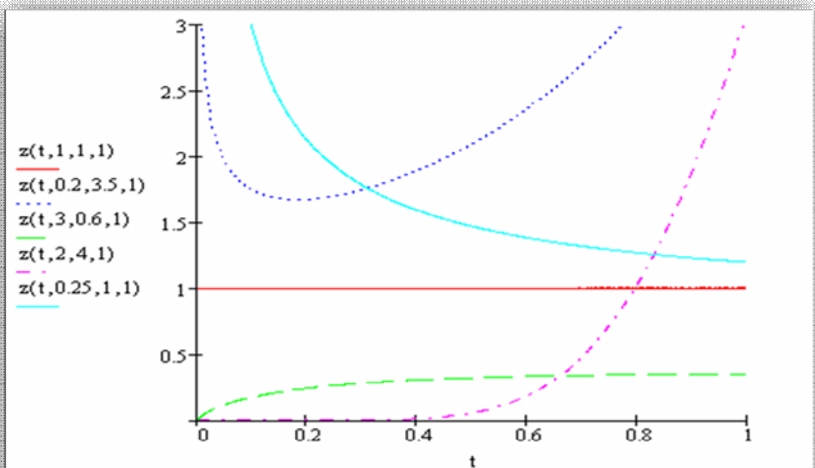

Figure (1): Failure rate function

One of the interesting properties of the EW distribution is that it can has different types of failure rate shapes. As it can be observed in Figure (1), when $\theta = \alpha = 1$ the FR function remains constant during the life time; if $\alpha \geq 1$ and $\alpha\theta \geq 1$ the FR function is monotone increasing; FR function is monotone decreasing for $\alpha \leq 1$ and $\alpha\theta \leq 1$; if $\alpha > 1$ and $\alpha\theta < 1$ we get the bathtub-shaped failure rate (BFR) function; upside-down bathtub-shaped failure rate (UBFRF) if $\alpha < 1$ and $\alpha\theta > 1$.

3- Mean Residual Life Function

The mean residual life (MRL) is defined as

$$MRL(t;\theta,\alpha,\lambda) = \frac{1}{1-[1-e^{-\left(\frac{t}{\lambda}\right)^{\alpha}}]^{\theta}} \int_t^{\infty} 1-[1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}]^{\theta}\, dx \quad (5)$$

For positive integer values of θ, we can write (5) in the form

$$MRL\,(t;\theta,\alpha,\lambda) = \theta \sum_{i=0}^{\theta-1} \frac{(-1)^i \binom{\theta-1}{i}}{(i+1)R(t)} \left\{ \lambda\Gamma\left(\frac{1}{\alpha}+1\right)\left[(i+1)^{-1/\alpha} - (i+1)^{-1}\Gamma\left((i+1)\left(\frac{t}{\lambda}\right)^{\alpha}\right)\left(\frac{1}{\alpha}\right)\right] + te^{-(i+1)\left(\frac{t}{\lambda}\right)^{\alpha}} \right\} - t \quad (6)$$

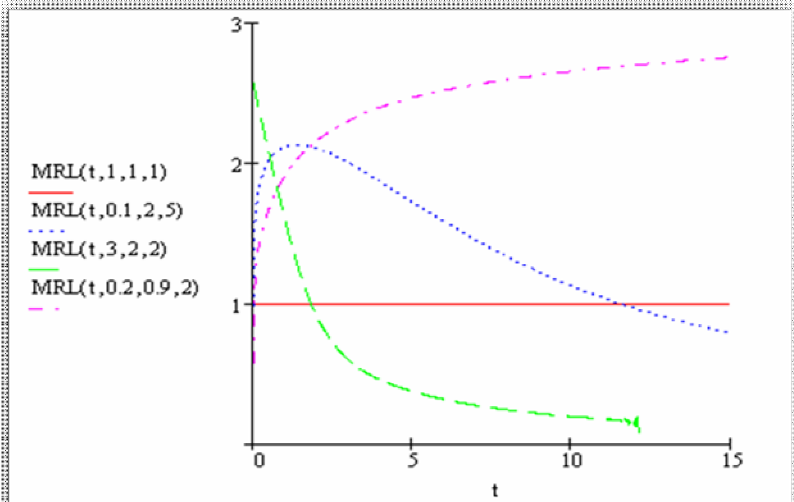

Figure (2): Mean residual life function

Figure (2) illustrates the shape of some mean residual life functions for EW where (θ, α) are the shape parameters and (λ) is a scale parameter. When $\theta = \alpha = 1$ the mean residual function remains constant during the life time; if $\alpha \geq 1$ and $\alpha\theta \geq 1$ the mean residual function is monotone decreasing; mean residual function is monotone increasing for $\alpha \leq 1$ and $\alpha\theta \leq 1$; if $\alpha > 1$ and $\alpha\theta < 1$ we get the BFR function; UBMRL if $\alpha > 1$ and $\alpha\theta < 1$.

4- Moments

For positive integer values of θ, the r^{th} non-central moment of the EW distribution is given by

$$E(T^r) = \lambda^r \Gamma(\frac{r}{\alpha}+1)\sum_{i=0}^{\theta-1}(-1)^i \binom{\theta-1}{i}(i+1)^{-\frac{r}{\alpha}-1} \qquad (7)$$

Special cases:

(i) The mean of the EW distribution is obtained by substituting $r = 1$ in (7), then the mean of EW distribution is given by

$$E(T) = \theta\lambda\, \Gamma(\frac{1}{\alpha}+1)\sum_{i=0}^{\theta-1}(-1)^i \binom{\theta-1}{i}(i+1)^{-\frac{1}{\alpha}-1} \qquad (8)$$

(ii) The second non-central moment of the EW distribution is given by

$$E(T^2) = \theta\lambda^2\, \Gamma(\frac{2}{\alpha}+1)\sum_{i=0}^{\theta-1}(-1)^i \binom{\theta-1}{i}(i+1)^{-\frac{2}{\alpha}-1} \qquad (9)$$

(iii) The third non-central moment is given by

$$E(T^3) = \theta\lambda^3\, \Gamma(\frac{3}{\alpha}+1)\sum_{i=0}^{\theta-1}(-1)^i \binom{\theta-1}{i}(i+1)^{-\frac{3}{\alpha}-1} \qquad (10)$$

We can find the variance of EW distribution

$$V(t) = E(T^2) - [E(T)]^2 \qquad (11)$$

5- Quantile

The quantile $Q(p)$ function of EW distribution (inverse cdf) can be shown as

$$Q(p) = F^{-1}(p) = \lambda[-ln(1-p^{1/\theta})]^{1/\alpha} \; ; \quad 0 < p < 1 \tag{12}$$

3 Estimation of the Parameters

In this section, we estimate the three parameters of EW distribution using different methods of estimation such as: maximum likelihood estimator (MLE), moment estimators (ME), percentile estimators (PCE) and least squares estimator (LSE).

3.1 Maximum Likelihood Estimators

Suppose that $T_1, T_2, \ldots, T_n$ are the n lifetimes observed arising from $f(t;\underline{\mu})$ in (2), where, $\underline{\mu} = (\theta, \alpha, \lambda)$, the likelihood function $L(\underline{\mu};t)$ is given by

$$L(\mu,t) = \theta^n \alpha^n \frac{1}{\lambda^{\alpha n}} \prod_{i=1}^{n} t_i^{\alpha-1} \, e^{-\sum_{i=1}^{n} (\frac{t_i}{\lambda})^{\alpha}} \prod_{i=1}^{n} [1 - e^{-(\frac{t_i}{\lambda})^{\alpha}}]^{\theta-1} \tag{13}$$

It is usually easer to maximize the natural logarithm of the likelihood function of both sides, so the natural logarithm of the likelihood function can be written as

$$\ln L(\mu;t) = nln(\theta) + nln(\alpha) - n\alpha \ln(\lambda) + (\alpha-1)\sum_{i}^{n} ln(t_i) - \sum_{i}^{n} \left(\frac{t_i}{\lambda}\right)^{\alpha} + (\theta-1)\sum_{i=1}^{n} \ln(1 - e^{-(\frac{t_i}{\lambda})^{\alpha}}) \tag{14}$$

The maximum likelihood estimates, $\hat{\mu} = (\hat{\theta}, \hat{\alpha}, \hat{\lambda})$ are obtained by taking the first derivative of the natural logarithm of the likelihood

function with respect to $\theta, \alpha \; and \; \lambda$ and equating to zero as follows

$$\frac{\partial \ln L(\underline{\mu}; t)}{\partial \theta} = \frac{n}{\hat{\theta}} + \sum_{i=1}^{n} \ln \left(1 - e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}\right) = 0 \tag{15}$$

$$\frac{\partial \ln L(\underline{\mu}; t)}{\partial \alpha} = \frac{n}{\hat{\alpha}} - n \ln(\hat{\lambda}) + \sum_{i=1}^{n} ln(t_i) - \frac{1}{\hat{\lambda}^{\hat{\alpha}}} \sum_{i=1}^{n} (t_i)^{\hat{\alpha}} \ln\left(\frac{t_i}{\hat{\lambda}}\right) + \frac{(\hat{\theta}-1)}{\hat{\lambda}^{\hat{\alpha}}} \sum_{i=1}^{n} \frac{e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}}{1-e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}} (t_i)^{\hat{\alpha}} \ln\left(\frac{t_i}{\hat{\lambda}}\right) = 0 \tag{16}$$

and

$$\frac{\partial \ln L(\underline{\mu}; t)}{\partial \lambda} = -\frac{n\hat{\alpha}}{\hat{\lambda}} + \frac{\hat{\alpha}}{\hat{\lambda}^{\hat{\alpha}+1}} \sum_{i=1}^{n} (t_i)^{\hat{\alpha}} + \frac{(\hat{\theta}-1)\hat{\alpha}}{\hat{\lambda}^{\hat{\alpha}+1}} \sum_{i=1}^{n} \frac{e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}}{1-e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}} (t_i)^{\hat{\alpha}} = 0 \tag{17}$$

From (15) we can get the MLE of θ as a function of $\hat{\alpha}$ and $\hat{\lambda}$:

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \ln\left(1-e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}\right)} \tag{18}$$

After some simplification, we have

$$-n + \frac{1}{\hat{\lambda}^{\hat{\alpha}}} \left[(\hat{\theta} - 1) \sum_{i=1}^{n} \frac{e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}}{1-e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}} t_i^{\hat{\alpha}} + \sum_{i=1}^{n} t_i \right] = 0 \tag{19}$$

and

$$\frac{n}{\hat{\alpha}} + \sum_{i=1}^{n} ln(t_i) - \frac{1}{\hat{\lambda}^{\hat{\alpha}}} \left[(\hat{\theta} - 1) \sum_{i=1}^{n} \frac{e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}}{1-e^{-\left(\frac{t_i}{\hat{\lambda}}\right)^{\hat{\alpha}}}} (t_i)^{\hat{\alpha}} \ln(t_i) + \sum_{i=1}^{n} (t_i)^{\hat{\alpha}} \ln(t_i) \right] = 0 \tag{20}$$

The system of the two non-linear equations (19) and (20) can be

solved with respect to α and $\boldsymbol{\lambda}$. Then we can use (18) together with

$\hat{\alpha}$ and $\hat{\lambda}$ to obtain the MLE of θ.

3.2 Moment Estimators

If T follows EW distribution with parameters $(\theta, \alpha, \lambda)$, then the first three non-central moments from the population of the EW are given in (8), (9) and (10) .

Similarly, the first three non-central moments from the random sample $T_1, T_2, \dots, T_n$ are

$$\bar{T} = \frac{\sum_{i=1}^{n} T_i}{n} \quad , \quad \frac{\sum_{i=1}^{n} T_i^2}{n} \quad \text{and} \quad \frac{\sum_{i=1}^{n} T_i^3}{n}$$

Therefore, equating the first three population moments with the corresponding sample moments, we have.

$$\bar{T} = \theta\lambda\, \Gamma\left(\frac{1}{\alpha}+1\right) \sum_{i=0}^{\theta-1} (-1)^i \binom{\theta-1}{i} (i+1)^{-\frac{1}{\alpha}-1} \qquad (21)$$

$$\frac{\sum_{i=1}^{n} T_i^2}{n} = \theta\lambda^2\, \Gamma\left(\frac{2}{\alpha}+1\right) \sum_{i=0}^{\theta-1} (-1)^i \binom{\theta-1}{i} (i+1)^{-\frac{2}{\alpha}-1} \qquad (22)$$

$$\frac{\sum_{i=1}^{n} T_i^3}{n} = \theta\lambda^3\, \Gamma\left(\frac{3}{\alpha}+1\right) \sum_{i=0}^{\theta-1} (-1)^i \binom{\theta-1}{i} (i+1)^{-\frac{3}{\alpha}-1} \qquad (23)$$

Then, the ME's of θ, α and λ, say $\tilde{\theta}, \tilde{\alpha}$ and $\tilde{\lambda}$, respectively, can be obtained by solving the three non-linear Equations (21), (22) and (23) numerically.

3.3 Percentile Estimators

If the data comes from a distribution function which has a closed form, then it is quite natural to estimate the unknown parameters by fitting a straight line to the theoretical points obtained from the distribution function and the sample percentile points. This method was originally explored by Kao in 1958, 1959 and it has been used quite successfully for Weibull distribution and for the generalized exponential distribution [see, Gupta and Kundu (2000)] and for the exponentiated Pareto distribution [see, Shawky and Abu-Zinadah (2009)].

In case of an EW distribution, it is possible to use the same concept to obtain the estimators of θ, α and λ based on the percentiles, since

$$F(t)=\left[1-e^{-\left(\frac{t}{\lambda}\right)^{\alpha}}\right]^{\theta}, \qquad \text{therefore,} \qquad \ln[F(t)]=\theta \ln\left[1-e^{-\left(\frac{t}{\lambda}\right)^{\alpha}}\right] \tag{24}$$

Let $T_{(i)}$ denotes the i^{th} order statistic, i.e. $T_1 < T_2 < \cdots < T_n$. If P_i denotes some estimate of $F(t_{(i)})$, then the percentile estimate of θ, α and λ can be obtained by minimizing

$$Q1 = \sum_{i=1}^{n} \left\{\ln[p_i] - \theta \ln\left[1-e^{-\left(\frac{t_{(i)}}{\lambda}\right)^{\alpha}}\right]\right\}^2 \tag{25}$$

with respect to θ, α and λ. It is possible to use several estimators of P_i as estimators of $F(t_{(i)})$. We mainly consider $P_i = \frac{i}{n+1}$ is the most used estimator as it is an unbiased estimator of $F(t_{(i)})$, i.e., $E\,[F(t_{(i)})] = P_i$. [See Castillo et al. (2005)].

To find the PCE, the first derivative of Q1 with respect to θ, α and λ is obtained then equating to zero:

$$\frac{\partial\, Q1}{\partial \theta} = \sum_{i=1}^{n} \left\{\ln[p_i] - \hat{\theta} \ln\left[1-e^{-\left(\frac{t_{(i)}}{\hat{\lambda}}\right)^{\hat{\alpha}}}\right]\right\} \ln\left[1-e^{-\left(\frac{t_{(i)}}{\hat{\lambda}}\right)^{\hat{\alpha}}}\right] = 0 \tag{26}$$

$$\frac{\partial\, Q1}{\partial \alpha} = \sum_{i=1}^{n} \left\{\ln[p_i] - \hat{\theta} \ln\left[1-e^{-\left(\frac{t_{(i)}}{\hat{\lambda}}\right)^{\hat{\alpha}}}\right]\right\} \frac{\hat{\theta}\, t_{(i)}^{\hat{\alpha}}}{\hat{\lambda}^{\hat{\alpha}}} \frac{e^{-\left(\frac{t_{(i)}}{\hat{\lambda}}\right)^{\hat{\alpha}}} \ln\left(\frac{t_{(i)}}{\hat{\lambda}}\right)}{1-e^{-\left(\frac{t_{(i)}}{\hat{\lambda}}\right)^{\hat{\alpha}}}} = 0 \tag{27}$$

and

$$\frac{\partial\, Q1}{\partial \lambda} = \sum_{i=1}^{n} \left\{\ln[p_i] - \hat{\theta} \ln\left[1-e^{-\left(\frac{t_{(i)}}{\hat{\lambda}}\right)^{\hat{\alpha}}}\right]\right\} \frac{\hat{\theta}\,\hat{\alpha}}{\hat{\lambda}^{\hat{\alpha}+1}} \frac{e^{-\left(\frac{t_{(i)}}{\hat{\lambda}}\right)^{\hat{\alpha}}}\, t_{(i)}^{\hat{\alpha}}}{1-e^{-\left(\frac{t_{(i)}}{\hat{\lambda}}\right)^{\hat{\alpha}}}} = 0 \tag{28}$$

Then, the PCE's of θ, α and $\boldsymbol{\lambda}$, say $\hat{\theta}_{PCE}, \hat{\alpha}_{PCE}$ and $\hat{\lambda}_{PCE}$, respectively,

can be obtained by solving the non-linear Equations (26), (27) and (28) numerically.

3.4 Least Squares Estimators

Suppose $T_1, T_2, \dots, T_n$ is a random sample of size n from a distribution function with cdf $G(.)$ and $T_{(1)} < \cdots < T_{(n)}$ denotes the order statistics of the observed sample. It is well known that

$$E\left(G(T_{(i)})\right) = \frac{i}{n+1}$$

The least squares estimators (LSE's) can be obtained by minimizing

$$Q2 = \sum_{i=1}^{n} \left\{G(T_{(i)}) - E\left(G(T_{(i)})\right)\right\}^2 \qquad (29)$$

with respect to the unknown parameters. Therefore, in case of EW distribution the least squares estimators of θ, α and λ, say $\hat{\theta}_{LSE}, \hat{\alpha}_{LSE}$ and $\hat{\lambda}_{LSE}$, respectively, can be obtained by minimizing

$$Q2 = \sum_{i=1}^{n} \left\{[1 - e^{-(\frac{t_{(i)}}{\lambda})^{\alpha}}]^{\theta} - \frac{i}{n+1}\right\}^2 \qquad (30)$$

with respect to θ, α and λ as follows:

$$\frac{\partial Q2}{\partial \theta} = \sum_{i=1}^{n} \left\{[1 - e^{-(\frac{t_{(i)}}{\hat{\lambda}})^{\hat{\alpha}}}]^{\hat{\theta}} - \frac{i}{n+1}\right\} [1 - e^{-(\frac{t_{(i)}}{\hat{\lambda}})^{\hat{\alpha}}}]^{\hat{\theta}} \, ln\left[1 - e^{-(\frac{t_{(i)}}{\hat{\lambda}})^{\hat{\alpha}}}\right] = 0 \quad (31)$$

$$\frac{\partial Q2}{\partial \alpha} = \sum_{i=1}^{n} \left\{[1 - e^{-(\frac{t_{(i)}}{\hat{\lambda}})^{\hat{\alpha}}}]^{\hat{\theta}} - \frac{i}{n+1}\right\} \hat{\theta}[1 - e^{-(\frac{t_{(i)}}{\hat{\lambda}})^{\hat{\alpha}}}]^{\hat{\theta}-1} e^{-(\frac{t_{(i)}}{\hat{\lambda}})^{\hat{\alpha}}} \left(\frac{t_{(i)}}{\hat{\lambda}}\right)^{\hat{\alpha}} \ln\left(\frac{t_{(i)}}{\hat{\lambda}}\right) = 0 \quad (32)$$

and

$$\frac{\partial Q2}{\partial \lambda} = \sum_{i=1}^{n} \left\{[1 - e^{-(\frac{t_{(i)}}{\hat{\lambda}})^{\hat{\alpha}}}]^{\hat{\theta}} - \frac{i}{n+1}\right\} \hat{\theta}[1 - e^{-(\frac{t_{(i)}}{\hat{\lambda}})^{\hat{\alpha}}}]^{\hat{\theta}-1} e^{-(\frac{t_{(i)}}{\hat{\lambda}})^{\hat{\alpha}}} \frac{\hat{\alpha}}{\hat{\lambda}^{\hat{\alpha}+1}} t_{(i)} = 0 \quad (33)$$

Then, the LSE's of the unknown parameters, say $\hat{\theta}_{LSE}$, $\hat{\alpha}_{LSE}$ and $\hat{\lambda}_{LSE}$, respectively, can be obtained by solving the three non-linear Equations (31), (32) and (33) numerically.

4 Simulation Study

It is very difficult to compare the theoretical performance of the different estimators described in the previous section; therefore, extensive Monte Carlo simulations are performed to compare the performance of the different methods of estimation using Mathcad (2001) software.

This section consists of two subsections. Subsection (4.1) compares the performance of the different estimators proposed in the present study. Subsection (4.2) selects the appropriate method of estimation by using total deviation method.

4.1 Comparative Study

The behavior of the different methods of estimation proposed for different sample sizes and different shape parameters is compared, mainly with respect to their relative absolute bias and relative absolute estimated risks (ER's). We consider different sample sizes from small to large. In all our computations we consider different values of the shape parameters θ, and α when $\lambda = 1$. We repeat each computation over N=1000 replications for different cases.

The algorithm for obtaining the different estimates by the different methods can be described in the following steps:

Step (1): Generate N=1000 random samples from the EW distribution. This can be achieved by firstly generating random samples from the uniform distribution. Then the uniform random numbers can be transformed to EW random numbers by using the following transformation

$$T_i = \lambda\left\{-\ln\left[1-(u_i)^{\frac{1}{\theta}}\right]\right\}^{\frac{1}{\alpha}} \;;\quad i = 1,2,\dots,n$$

Step (2): We consider the estimation of the parameters, can be obtained by solving the non-linear equations in Section (3).

Step (3): The previous steps are repeated 1000 times. The average of estimates are computed by averaging the estimate of the parameters $\underline{\mu} = (\theta, \alpha, \lambda)$, as follows

$$\underline{\bar{\bar{\mu}}_j} = \frac{\sum_{i=1}^{N} \bar{\bar{\mu}}_{ij}}{N}, \qquad j=1, 2, 3$$

where $\bar{\bar{\mu}}_j$ denote the average of the estimates of μ_j.

The variance of the estimates is obtained as follows

$$Var(\bar{\hat{\mu}}_j) = \frac{1}{N}\left[\sum_{i=1}^{N}[\hat{\mu}_{ij} - \bar{\hat{\mu}}_j]^2\right], \qquad j=1,2,3$$

The bias of the estimates are obtained by

$$Bias(\bar{\hat{\mu}}_j) = Var(\bar{\hat{\mu}}_j) - \mu_j, \quad j = 1,2,3$$

We can obtain the relative absolute bias as

$$Rbias(\bar{\hat{\mu}}_j) = \left|\frac{Bias(\bar{\hat{\mu}}_j)}{\mu_j}\right|, \qquad j=1,2,3$$

The estimated mean squared errors or shortly the estimated risks (ER's) is computed as

$$ER(\bar{\hat{\mu}}_j) = Var(\bar{\hat{\mu}}_j) + [Bias(\bar{\hat{\mu}}_j)]^2, \quad j = 1,2,3$$

Then, we can obtain the relative absolute estimated risks by using

$$RER(\bar{\hat{\mu}}_j) = \left|\frac{ER(\bar{\hat{\mu}}_j)}{\mu_j}\right|, \qquad j=1,2,3$$

Step (4): The results of the estimates of the parameters $\hat{\theta}, \hat{\alpha}, \hat{\lambda}$ and their variances (Var.`s), relative absolute bias and relative absolute ER. are illustrated in Tables (1) to (4) in Appendix A.

From these tables the following conclusions can be noticed on the performance of the methods of estimation:

- The variances and the relative absolute estimated risks are decreasing as the sample size increases for different shape parameters θ, α for all methods.
- The variances and RER of MLE`s are less than the variances and RER of other estimators.

- The relative bias of ME`s is less than the relative bias of other methods.

4.2 Selecting the appropriate method of estimation

The criterion of selecting the appropriate method of estimation is the total deviation (TD) and the appropriate one which gives the minimum TD.

To compare, we calculate the TD for each method as follows:

$$TD = \left|\frac{\hat{\theta}-\theta}{\theta}\right| + \left|\frac{\hat{\alpha}-\alpha}{\alpha}\right| + \left|\frac{\hat{\lambda}-\lambda}{\lambda}\right| \qquad (34)$$

where θ, α and λ are the parameters of the distribution, $\hat{\theta}, \hat{\alpha}$ and $\hat{\lambda}$ are the estimates of these parameters by using any method. [See Al-Fawzan (2000)].

Table (5) shows the results when $\alpha = 2, \lambda = 1$ for different values of the shape parameter θ. Table (6) displays the results when $\theta = 2, \lambda = 1$ and different values of the shape parameter α. We can choose the appropriate method which yields the minimum TD. Notice, that the maximum TD is 4.260 from PCE, and the minimum TD is 0.276 from ME for all methods. [See Table (5) and (6) in Appendix B].

5 Sampling Distribution

In this section, the sampling distributions of ME (the appropriate method of estimating the parameters) are derived using Pearson's systems. The Pearson's system contains seven basic types of distribution together in a single parametric framework. The selection approach is based on a certain quantity, K, which is a function of the first four non-central central moments, that is:

$$K = \frac{\beta_1 (\beta_2+3)^2}{4\,(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)}, \tag{35}$$

where β_1 and β_2 denote the skewness and kurtosis measures respectively. So for different values of k, there exist different types of Pearson curves. If $K < 0$, we fit Type I Pearson curve, while Type II is fitted if $K = 0$ and $\beta_2 > 3$. If $K = \infty$ and

$2\beta_2 - 3\beta_1 - 6 = 0$, we obtain Type III. We get Type IV If $0 < K < 1$. When $K = 1$, Type V is obtained. If $K > 1$, get Type VI. Finally, If $K = 0$ and $\beta_2 < 3$ Type VII is obtained. [See Elderton and Johnson (1969)].

The sampling distributions of the ME for each value of the parameter and for different sample sizes are displayed in Tables (7) to (10). [See Appendix C].

As a result of computer simulation; three sampling distributions were fitted to the ME which are Pearson Type I, IV and VI distributions

6 Conclusion

The EW distribution is flexible in modeling various types of failure data with possibly increasing, constant, decreasing or bathtub shaped FR function. Also the mean residual life function has various types with possibly increasing, constant, decreasing or upside-down bathtub shaped MRL function.

In this paper, a three-parameter EW distribution is discussed and its main properties and the r^{th} non-central moments are derived. Four different methods of estimation are discussed and a comparison is made according to different criteria. The sampling distributions of the estimates of the appropriate method of estimation are derived.

The simulation study indicates that: the variances and relative absolute estimate of the unknown parameters for most of the methods decrease as the sample sizes increase. The variances and RER of MLE`s are less than the variances and RER of other estimators. The relative bias of ME`s is less than the relative bias of other methods.

It is found that using the total deviation that the method of estimate which gives the appropriate estimates is the ME. In addition, the sampling distributions of ME are obtained, it is found that most of the sampling distributions for different values of parameters under different sample sizes follow Pearson's Type *I* ,*IV* and *VI* distributions.

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APPENDIX A

Results of Estimating the Parameters

Table (1)

Mean of estimates, with their variance, relative absolute bias and relative absolute estimated risk. Population parameter values: $\theta = 0.5, \alpha = 2, \lambda = 1$

n= 10					$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER		Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	1.051	0.178	1.101	0.962	1.373	0.196	0.313	0.294	0.68	0.024	0.320	0.127
ME	1.138	0.395	1.276	1.604	1.601	0.182	0.199	0.171	0.691	0.035	0.309	0.131
PCE	0.633	0.593 $\hat{\theta}$	0.266	1.222	1.779 $\bar{\hat{\alpha}}$	24.796	1.889	19.537	3.103	82.205	2.103	86.629
LSE	0.804	4.355	0.608	8.896	0.109	30.076	0.946	16.826	0.787	2.949	0.213	2.995
n= 20	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	1.024	0.081	1.048	0.712	1.308	0.089	0.346	0.284	0.675	0.011	0.325	0.117
ME	1.081	0.236	1.162	1.148	1.512	0.091	0.244	0.165	0.692	0.021	0.308	0.116
PCE	0.604	0.519	0.208	1.059	1.862	18.093	0.068	19.093	2.856	50.778	1.856	54.222
LSE	0.767	3.500	0.534	7.142	1.249	15.990	0.375	8.277	0.793	2.475	0.207	0.042
n= 30	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	1.015	0.050	1.030	0.630	1.283	0.057	0.358	0.285	0.673	0.0072	0.327	0.114
ME	1.051	0.150	1.102	0.908	1.474	0.058	0.263	0.167	0.693	0.014	0.307	0.109
PCE	0.600	0.510	0.200	1.040	1.870	12.007	0.065	6.011	2.549	30.948	1.549	33.347
LSE	0.684	2.080	0.368	4.227	1.586	10.777	0.207	5.474	0.826	2.300	0.174	2.330
n= 50	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	1.008	0.026	1.015	0.567	1.265	0.030	0.368	0.285	0.670	0.0038	0.330	0.113
ME	1.030	0.089	1.06	0.74	1.452	0.034	0.274	0.167	0.695	0.0084	0.305	0.102
PCE	0.559	0.436	0.118	0.878	1.986	6.882	0.007	3.441	2.351	12.074	1.351	13.899
LSE	0.608	1.887	0.216	3.797	1.660	4.537	0.170	2.326	0.830	2.009	0.170	2.037

Table (2)

Mean of estimates, with their variance, relative absolute bias and relative absolute estimated risk. Population parameter values: $\theta = 1.5, \alpha = 2, \lambda = 1$

n= 10	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.968	0.073	0.355	0.238	2.337	0.403	0.169	0.258	1.305	0.083	0.305	0.176
ME	1.813	1.778	0.209	1.251	2.49	0.801	0.245	0.52	1.037	0.078	0.037	0.079
PCE	2.121	20.083	0.414	13.645	1.416	0.157	0.292	0.249	0.711	0.178	0.290	0.262
LSE	2.944	10.339	0.962	8.282	1.445	4.342	0.277	2.325	1.508	682.407	0.508	682.666
n= 20	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.987	0.039	0.342	0.202	2.252	0.181	0.126	0.122	1.281	0.038	0.281	0.117
ME	1.942	1.939	0.295	1.423	2.291	0.565	0.145	0.325	1.005	0.075	0.005	0.075
PCE	2.365	2.386	0.576	2.089	1.382	0.154	0.309	0.267	1.008	0.128	0.008	0.128
LSE	2.332	6.761	0.554	4.968	1.698	4.643	0.151	2.367	1.400	50.157	0.400	50.317
n= 30	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.993	0.025	0.338	0.188	2.211	0.098	0.106	0.071	1.275	0.025	0.275	0.101
ME	1.927	1.922	0.285	1.403	2.225	0.442	0.113	0.247	1.004	0.070	0.004	0.07
PCE	2.036	2.310	0.357	1.731	1.489	0.144	0.255	0.202	1.009	0.107	0.009	0.107
LSE	2.133	4.318	0.422	3.145	1.715	4.557	0.142	2.319	1.388	20.531	0.388	20.681
n= 50	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.995	0.014	0.337	0.179	2.188	0.06	0.094	0.048	1.269	0.014	0.269	0.087
ME	1.909	1.901	0.273	1.379	2.195	0.375	0.098	0.207	1.006	0.068	0.003	0.068
PCE	2.002	2.239	0.334	1.660	1.490	0.099	0.255	0.179	1.005	0.100	0.005	0.100
LSE	2.125	4.005	0.416	2.930	1.750	4.128	0.125	2.095	1.326	10.882	0.326	10.988

Table (3)

Mean of estimates, with their variance, relative absolute bias and relative absolute estimated risk. Population parameter values: $\theta = 2, \alpha = 2, \lambda = 1$

n= 10	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.936	0.045	0.532	0.589	2.292	0.349	0.146	0.217	1.53	0.103	0.530	0.384
ME	2.182	2.149	0.091	1.091	2.670	1.123	0.335	0.786	1.083	0.085	0.083	0.092
PCE	1.468	7.12	0.266	3.701	2.680	6.971	0.340	3.717	0.369	0.089	0.631	0.487
LSE	2.701	17.164	0.350	8.828	2.291	8.459	0.145	4.272	0.551	0.50	0.449	0.702
n= 20	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.972	0.025	0.514	0.541	2.207	0.149	0.104	0.096	1.493	0.046	0.493	0.290
ME	2.146	2.125	0.073	1.073	2.522	0.796	0.261	0.534	1.081	0.083	0.081	0.089
PCE	1.181	4.352	0.409	2.511	2.237	4.04	0.118	4.096	0.340	0.066	0.660	0.501
LSE	2.609	16.266	0.304	8.319	2.233	8.206	0.116	4.130	0.501	0.402	0.499	0.651
n= 30	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.982	0.017	0.509	0.526	2.173	0.086	0.086	0.058	1.484	0.030	0.484	0.265
ME	2.266	2.195	0.132	1.133	2.391	0.640	0.195	0.396	1.056	0.081	0.056	0.085
PCE	1.168	0.974	0.416	1.666	2.119	2.424	0.059	1.219	0.309	0.050	0.691	0.527
LSE	2.398	1.388	0.199	0.773	2.088	1.353	0.044	0.680	0.751	0.048	0.249	0.110
n= 50	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.989	0.009	0.505	0.515	2.15	0.052	0.075	0.037	1.476	0.017	0.476	0.244
ME	2.170	2.141	0.085	1.085	2.386	0.550	0.193	0.350	1.071	0.077	0.071	0.082
PCE	1.506	0.777	0.247	0.510	2.080	1.997	0.04	1.001	0.508	0.060	0.492	0.302
LSE	2.229	1.003	0.114	0.527	2.100	1.109	0.050	0.559	0.888	0.050	0.112	0.062

Table (4)

Mean of estimates, with their variance, relative absolute bias and relative absolute estimated risk. Population parameter values: parameters $\theta = 2, \alpha = 1.5, \lambda = 1$

n= 10	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.925	0.043	0.537	0.599	1.721	0.196	0.148	0.164	1.778	0.253	0.778	0.858
ME	1.642	1.514	0.179	0.821	2.226	0.624	0.484	0.767	1.259	0.129	0.259	0.196
PCE	0.038	0.040	0.981	1.945	2.303	2.593	0.535	2.158	0.107	0.116	0.893	0.913
LSE	1.598	13.209	0.201	6.685	2.072	43.423	0.381	29.167	0.682	3.591	0.318	3.692
n= 20	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.967	0.025	0.517	0.546	1.657	0.084	0.104	0.072	1.714	0.110	0.714	0.620
ME	1.768	1.714	0.116	0.884	2.014	0.397	0.343	0.441	1.218	0.120	0.218	0.168
PCE	0.608	0.129	0.696	1.034	2.108	0.527	0.405	0.598	0.051	0.0005	0.949	0.901
LSE	1.653	9.740	0.173	4.930	1.995	38.510	0.330	25.836	0.693	3.150	0.307	3.244
n= 30	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.979	0.017	0.511	0.530	1.630	0.048	0.087	0.044	1.698	0.071	0.698	0.558
ME	1.885	1.872	0.058	0.942	1.912	0.328	0.274	0.331	1.183	0.122	0.183	0.155
PCE	0.625	0.137	0.687	1.013	2.177	0.965	0.451	0.949	0.048	0.0004	0.952	0.908
LSE	1.678	5.998	0.161	3.050	1.827	25.776	0.216	17.255	0.699	2.555	0.303	2.644
n= 50	$\hat{\theta}$				$\hat{\alpha}$				$\hat{\lambda}$			
	$\bar{\hat{\theta}}$	Var.	Rbias	RER	$\bar{\hat{\alpha}}$	Var.	Rbias	RER	$\bar{\hat{\lambda}}$	Var.	Rbias	RER
MLE	0.987	0.009	0.506	0.517	1.613	0.029	0.076	0.028	1.684	0.039	0.684	0.507

ME	1.927	1.922	0.036	0.963	1.859	0.282	0.240	0.274	1.171	0.123	0.171	0.152
PCE	0.809	0.071	0.595	0.744	1.965	0.088	0.310	0.202	0.032	0.0001	0.968	0.937
LSE	1.712	2.104	0.144	1.095	1.774	10.002	0.183	6.716	0.722	1.555	0.278	1.632

APPENDIX B

Total Deviation Method

Table (5)

Comparison between different methods of estimation with different θ

$(\alpha = 2, \lambda = 1)$

θ	α	λ	Size	MLE	MME	PCE	LSE	Best
				TD	TD	TD	TD	
0.5	2	1	10	1.735	1.784	4.260	1.767	MLE
			20	1.719	1.714	2.133	1.116	LSE
			30	1.715	1.672	1.814	0.749	LSE
			50	1.713	1.639	1.476	0.556	LSE
1.5	2	1	10	0.829	0.491	0.995	1.748	ME
			20	0.748	0.445	0.893	1.105	ME
			30	0.719	0.401	0.621	0.952	ME
			50	0.700	0.377	0.594	0.867	ME

2	2	1	10	1.208	0.509	1.237	0.945	ME
			20	1.110	0.415	1.188	0.920	ME
			30	1.079	0.385	1.166	0.492	ME
			50	1.056	0.349	0.779	0.276	LSE

Table (6)

Comparison between different methods of estimation with different α

$(\theta = 2, \lambda = 1)$

α	θ	λ	Size	MLE	MME	PCE	LSE	Best
				TD	TD	TD	TD	
1.5	2	1	10	1.463	0.922	2.409	0.901	LSE
			20	1.335	0.677	2.051	0.810	ME
			30	1.295	0.515	2.091	0.680	ME
			50	1.266	0.447	1.873	0.605	ME
2	2	1	10	1.208	0.509	1.237	0.945	ME
			20	1.110	0.415	1.188	0.920	ME
			30	1.079	0.385	1.166	0.492	ME

			50	1.056	0.349	0.779	0.276	LSE

APPENDIX C

Sampling Distribution of ME

Table (7)

Sampling distribution of the ME of the parameters

$\theta = 0.5, \alpha = 2, \lambda = 1$

Sample size n	Parameters	Skewness	Kurtosis	Person's Coefficient	Type
10	θ	4.334	19.787	0.414	IV
	α	1.007	5.366	0.559	IV
	λ	0.073	3.193	0.335	IV
20	θ	5.837	35.065	0.369	IV
	α	0.434	4.017	0.493	IV
	λ	-0.386	4.433	-0.070	I

30	θ	7.473	56.841	0.383	IV
	α	0.167	3.896	0.102	IV
	λ	-0.683	5.585	-0.072	I
50	θ	9.849	98.01	0.432	IV
	α	-0.091	4.201	-0.026	I
	λ	-0.999	7.788	-0.068	I

Table (8)

Sampling distribution of the ME of the parameters

$\boldsymbol{\theta = 1.5, \alpha = 2, \lambda = 1}$

Sample size n	Parameters	Skewness	Kurtosis	Person's Coefficient	Type
10	θ	1.030	2.062	-0.258	I
	α	0.832	4.867	0.612	IV
	λ	-0.486	2.348	-2.102	I
20	θ	0.802	1.642	-0.203	I
	α	0.258	2.682	-0.148	I
	λ	-0.590	2.003	1.673	VI

30	θ	0.974	1.950	-0.244	I
	α	-0.045	2.149	0.022	IV
	λ	-0.631	1.912	1.415	VI
50	θ	0.857	1.735	-0.216	I
	α	-0.352	1.932	0.226	IV
	λ	-0.74	1.863	8.531	VI

Table (9)

Sampling distribution of the ME of the parameters

$\boldsymbol{\theta = 2, \alpha = 2, \lambda = 1}$

Sample size n	Parameters	Skewness	Kurtosis	Pearson's Coefficient	Type
10	θ	0.438	1.192	-0.113	I
	α	0.968	4.928	0.949	IV
	λ	-0.218	1.951	0.109	IV
20	θ	0.486	1.236	-0.125	I
	α	0.402	2.642	-0.178	I
	λ	-0.369	1.659	0.164	IV
	θ	0.316	1.100	-0.081	I

30					
	α	0.192	1.898	-0.059	I
	λ	-0.251	1.427	0.079	IV
50	θ	0.832	1.692	-0.210	I
	α	-0.313	1.974	0.197	IV
	λ	-0.736	1.824	3.138	VI

Table (10)

Sampling distribution of the ME of the parameters

$\alpha = 1.5, \theta = 2, \lambda = 1$

Sample size n	Parameters	Skewness	Kurtosis	Pearson's Coefficient	Type
10	θ	1.395	2.945	-0.378	I
	α	0.814	4.824	0.613	IV
	λ	-0.539	2.693	-0.352	I
20	θ	1.118	2.250	-0.281	I
	α	0.283	2.938	-0.235	I
	λ	-0.753	2.405	-0.433	I
30	θ	0.899	1.808	-0.225	I

	α	9.653×10^{-5}	2.18	-4.529×10^{-5}	I
	λ	-0.65	1.965	3.440	VI
50	θ	0.827	1.683	-0.208	I
	α	-0.243	1.889	0.117	IV
	λ	-0.691	1.788	1.228	VI