An Inter-Arrival Hyper-Exponential Machine Repairman problem with Two Heterogeneous Repairmen

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Abstract- This paper aims to treat the analytical solution of the truncated inter-arrival hyper-exponential machine interference queue : $H_r/M/2/k/N$ in case of r branches with two heterogeneous repairmen . Our research treats the general case for the value of r, k and N considering the discipline FIFO with a more general condition . In the end we study some special cases.

Keywords: Heterogeneous repairmen; machine interference; inter-arrival hyper-exponential

1- **Introduction :**

The truncated inter-arrival hyper-exponential machine interference queue: $H_r/M/c/k/N$ with balking and reneging treated by Shawky [9], George et al. [3] considered multi-server system of the type $GI/M/r$ where the coefficient of variation of inter-arrival times is greater than 1, and derived two simple approximations for the steady state average queueing time. Gupta [5] treated numerically the inter-arrival hyper-exponential queue: $H_r/M/1/m$ with state dependent arrival and service rates, and Abou-AlAta [1] treated the analytical solution of the truncated inter-arrival hyper-exponential machine interference queue : $H_2/M/c/m/m$ with both balking and reneging in case homogeneous servers. The present paper treats the analytical solution of the truncated inter- arrival hyper-exponential machine interference queue: $H_r/M/2/k/N$ with two heterogeneous repairmen using a recurrence relation. Also some special cases are obtained.

2 - Model Description:

As in Goyal [4] **,** the arrival channel consists of r independent branches. A machine arriving for service joins the i^{th} branch with a fraction σ_i of the time on the average, so that $\sum_{i=1}^{r} \sigma_i = 1$. Only one machine can occupy any one of the branches at a time and if a machine is present in any one of the branches, the arrival channel is busy and no other machine can enter any other branch. The machine in the i^{th}branch joins the system (queue or service) with rate λ_i per unit time. Finite source (population) of N machines, finite space with capacity k machines ($k < N$), and two heterogeneous servers(repairmen) are assumed. The service times are identical exponential random variables with ratesμ_1, and μ_2 ($\mu_1 > \mu_2$).

An arriving machine may find :

(i) All repairmen engage, then it waits in a line in order of arrival, the machine at the top of the queue occupied the repairman that falls vacant first.

(ii) Only one of the repairmen free , so it occupies the free repairman.

(iii) All repairmen free, thus it chooses repairman I with probability π_1 , repairman II with probability π_2, such that, $\pi_1 + \pi_2 = 1$. The classical queue discipline assumes that $\pi_1 = \pi_2 = \frac{1}{2}$. Saaty [12] took π_1 to be $\mu_1(\mu_1 + \mu_2)^{-1}$ and π_2 to be $\mu_2(\mu_1 + \mu_2)^{-1}$. Gumble [6] has studied the many server (heterogeneous) queue but under the classical queue discipline and Krishnamoorthi [8] has studied Poisson- queue with two heterogeneous servers with different queue disciplines.

3-The Steady- State equations and their solution

Define the equilibrium probabilities:

$p_{0,0,s}$= prob. {there is no machine in the system and s^{th} arrival branch occupied the next arriving machine},

$p_{1,0,s}$= prob. {there is one machine in repairman $I$and s^{th} arrival branch occupied the next arriving machine},

$p_{0,1,s}$= prob. {there is one machine in repairman $II$and s^{th} arrival branch occupied the next arriving machine},

$p_{n,s}$= prob. {there are n machines in the system and s^{th} arrival branch occupied the next arriving machine} , n=2.,3,..k; s =1,2,… r .

Also, $p_{0,s} = p_{0,0,s}$, $p_{1,s} = p_{1,0,s} + p_{0,1,s}$ and $p_{2,s} = p_{1,1,s}$.

Consequently, the steady-state probability difference equations are:

$$N\,\lambda_i p_{0,0,i} = \mu_1 p_{1,0,i} + \mu_2 p_{0,1,i} \quad (1)$$

$$\{(N-1)\lambda_i + \mu_1\}p_{1,0,i} = \sigma_i \pi_1 \sum_{s=1}^{r} N\,\lambda_s p_{0,s} + \mu_2 p_{1,1,i} \quad ,i = 1(1)r \quad (2)$$

$$\{(N-1)\lambda_i + \mu_2\}p_{0,1,i} = \sigma_i \pi_2 \sum_{s=1}^{r} N\,\lambda_s p_{0,s} + \mu_1 p_{1,1,i} \quad ,i = 2(1)r \quad (3)$$

$$\{(N-n)\lambda_i + \mu\}p_{n,,i} = (N-n+1)\sigma_i \sum_{s=1}^{r} \lambda_s p_{n-1,s} + \mu\, p_{n+1,i}, n = 2,3,..,k-1 \quad (4)$$

$$\{(N-k)\lambda_i + \mu\}p_{k,i} = (N-k+1)\sigma_i \sum_{s=1}^{r} \lambda_s p_{k-1,s} + (N-k)\sigma_i \sum_{s=1}^{r} \lambda_s p_{k,s} \quad (5)$$

where : $\mu = \mu_1 + \mu_2$.

Summing up equations (1) –(5)over i, and adding the results obtained for $n = 1,2, \dots k-1$, we get

$$\sum_{s=1}^{r} p_{n+1,s} = (N-n+1)\sum_{s=1}^{r} \rho_s\, p_{n,s} \ , \ n = 1,2, \dots, k-1 \quad (6)$$

where $\rho_s = \frac{\lambda_s}{\mu}$.

From (5) and (6) for $n = k-1$, we have

$$[(N-k)\rho_i + 1]p_{k,i} = \sigma_i \sum_{s=1}^{r} [(N-k)\rho_s + 1]\, p_{k,s} \,, \quad (7)$$

It is easy to verify that the determinate formed by the coefficients of $p_{k,i}, i = 1,2,..r$ is zero and therefore we can solve equation(7) for any $r-1$ probabilities involved in terms of $p_{k,r}$. Leaving out the r^{th} equation, we have the matrix representation of (7) as

$$\boldsymbol{C}\,\boldsymbol{R} = -\,[(N-k)\rho_r + 1]\mathbf{G}p_{k,r} ,$$

where $\boldsymbol{C}$ is the $(r-1)\times(r-1)$ matrix

$$\boldsymbol{C} = [c_{ij}]$$

such that

$$c_{ij} = \sigma_i\big[(N-k)\rho_j + 1\big], i \neq \mathrm{j}$$

$$c_{ii} = (\sigma_i - 1)[(N-k)\rho_i + 1],$$

where

$$R^T = \big[p_{k,1}, p_{k,2}, ..., p_{k,r-1}\big],$$

$$\mathbf{G}^T = [\sigma_1, \sigma_2, \dots, \sigma_{r-1}].$$

Now, C^{-1} is given by

$$C^{-1} = [c_{ij}^*]$$

such that

$$c_{ij}^* = \frac{-\sigma_i}{\sigma_r[(N-k)\rho_i + 1]}, i \neq j,$$

$$c_{ii}^* = \frac{-(\sigma_i+\sigma_r)}{\sigma_r[(N-k)\rho_i+1]},$$

and

$$p_{k,i} = \frac{\sigma_i[(N-k)\rho_r+1]}{\sigma_r[(N-k)\rho_i+1]}\, p_{k,r}, i = 1,2,\dots,r-1\,. \tag{8}$$

From (4) and (6) we obtain

$$\{(N-n)\rho_i - \sigma_i + 1\}p_{n,,i} - \sigma_i \textstyle\sum_{s\neq i}^{r} p_{n,s} = \mathrm{p}_{n+1,i}, n = 2,3, \dots k-1; i = 1,2,\dots,r \tag{9}$$

which can be written in the matrix form as:

$$A\,P = B \tag{10}$$

where

$$A = [a_{ij}]$$

such that

$$a_{ij} = -\sigma_i, i \neq j$$

$$a_{ii} = \{(N-n)\rho_i - \sigma_i + 1\},$$

$$P^T = [p_{n,1}, p_{n,2}, \dots, p_{n,r}],$$

and

$$B^T = \left[p_{n+1,1}, p_{n+1,2}, \dots, p_{n+1,r}\right],$$

where T denotes the transpose of a matrix. Now, A^{-1} is given by

$$A^{-1} = [a_{ij}^*],$$

where

$$a_{ij}^* = \frac{\sigma_i}{D_n\,[(N-n)\rho_i+1]\;[(N-n)\rho_j+1]}, i \neq j,$$

$$a_{ii}^* = \frac{1}{[(N-n)\rho_i+1]} + \frac{\sigma_i}{D_n\,[(N-n)\rho_i+1]^2},$$

and $$D_n = 1 - \sum_{i=1}^{r} \frac{\sigma_i}{[(N-n)\rho_i+1]},\; n = 2,3,\dots,k-1.$$

Using this value of A^{-1} in (10), we get

$$p_{n,i} = \left[\frac{p_{n+1,i}}{[(N-n)\rho_i+1]} + \frac{\sigma_i}{D_n\,[(N-n)\rho_i+1]}\sum_{s=1}^{r} \frac{p_{n+1,s}}{[(N-n)\rho_s+1]}\right],$$

$$n = k-1, k-2, \dots, 2; i = 1,2,\dots,r \quad (11)$$

From (1)-(3),we have

$$p_{1,0,i} = l_i\eta + g_i\, p_{1,1,i}\,; \qquad i = 1(1)r\ ,$$

$$p_{0,1,i} = e_i\eta + f_i p_{1,1,i}\,; \qquad i = 2(1)r\ ,$$

$$p_{0,0,i} = \frac{\mu_1}{N\,\lambda_i} p_{1,0,i} + \frac{\mu_2}{N\,\lambda_i} p_{0,1,i};\; i = 1(1)r,$$

where

$$g_i = \frac{\mu_2}{(N-1)\lambda_i+\mu_1}, \quad l_i = \frac{\sigma_i\pi_1}{(N-1)\lambda_i+\mu_1},$$

$$f_i = \frac{\mu_1}{(N-1)\lambda_i+\mu_2}\ , \quad e_i = \frac{\sigma_i\pi_2}{(N-1)\lambda_i+\mu_2}\ ,$$

and

$$\eta = \frac{\sum_{s=1}^{r}(\mu_1 g_s + \mu_2 f_{s)}\, p_{2,s}}{1-\{\mu_1\ \sum_{s=1}^{r} l_s + \mu_2 \sum_{s=1}^{r} e_s\}}\,.$$

Thus, we have expressed all probabilities for $n = 0,1,2,\dots,k; i = 1,2,\dots,r$ in terms of one unknown probability, namely $p_{k,r}$. This unknown probability may now be computed by using the normalizing condition: $\sum_{n=0}^{k}\sum_{i=1}^{r} p_{n,i} = 1,$

and hence all the probabilities are completely known in terms of the queue parameters.

4-The main performance measures characteristic of machine interference problem are mentioned:

1-The expected number of down machines is

$$L = \sum_{n=1}^{k} \sum_{s=1}^{r} n\, p_{n,s}\ .$$

2-Machine efficiency (the fraction of total production time on all machines) is

$$U_m = 1 - \frac{L}{k}.$$

3-Average operator utilization

$$U_s = \sum_{n=0}^{k} \sum_{s=1}^{r} \frac{n\, p_{n,s}}{c} + \sum_{n=c+1}^{k} \sum_{s=1}^{r} p_{n,s}$$

or

$$U_s = 1 - \sum_{n=0}^{k} \sum_{s=1}^{r} (1 - \tfrac{n}{c}) p_{n,s}.$$

5-Numerical Work

The following example illustrates the method discussed above.

Example: In the system: $H_r\ /M/2/k/N$ letting r =2, k =3 and $N = 4$, i.e., the queue : $H_2\ /M/2/3/4$, the results are:

$$p_{3,1} = a\, p_{3,2}, \qquad p_{2,1} = b_1\, p_{3,2}\ , p_{2,2} = b_2\, p_{3,2},$$

$$p_{0,1,1} = e_1\, p_{3,2}, \qquad p_{0,1,2} = e_2\, p_{3,2},$$

$$p_{1,0,1} = d_1\, p_{3,2}, \qquad p_{1,0,2} = d_2\, p_{3,2}$$

$$p_{0,0,1} = h_1\, p_{3,2}, \quad p_{0,0,2} = h_2\, p_{3,2},$$

$$\eta = \gamma\, p_{3,2}, \quad \gamma = \frac{(\mu_1 g_1 + \mu_2 f_1)b_1 + (\mu_1 g_2 + \mu_2 f_2)b_2}{1 - \{\mu_1 (l_1 + l_2) + \mu_2 (l_1 + l_2)\}},$$

where

$$g_i = \frac{\mu_2}{3\lambda_i + \mu_1}, \qquad l_i = \frac{\sigma_i \pi_1}{3\lambda_i + \mu_1}, i = 1,2$$

$$f_i = \frac{\mu_1}{3\lambda_i + \mu_2}\ , \quad e_i = \frac{\sigma_i \pi_2}{3\lambda_i + \mu_2}\ ; i = 1,2,$$

$$a = \frac{\sigma_1(\rho_2 + 1)}{\sigma_2(\rho_1 + 1)}, \quad b_1 = \frac{1}{2\rho_1 + 1}\left[a + \frac{\sigma_1}{D_2}\left(\frac{a}{2\rho_1 + 1} + \frac{1}{2\rho_2 + 1}\right)\right],$$

$$b_2 = \frac{1}{2\rho_2 + 1}\left[a + \frac{\sigma_2}{D_2}\left(\frac{a}{2\rho_1 + 1} + \frac{1}{2\rho_2 + 1}\right)\right],$$

$$d_1 = (g_1 b_1 + l_1\,\gamma), \quad d_2 = (g_2 b_2 + l_2\,\gamma),$$

$$c_1 = (f_1 b_1 + e_1\,\gamma), \quad c_2 = (f_2 b_2 + e_2\,\gamma),$$

$$h_1 = \frac{\mu_1 d_1 + \mu_2\, e_1}{4\,\lambda_1}, \qquad h_2 = \frac{\mu_1 d_2 + \mu_2\, c_2}{4\,\lambda_2}.$$

From the normalizing condition: $\sum_{n=0}^{3}\sum_{i=1}^{2} p_{n,i} = 1$, we have

$$p_{3,2} = [1 + a + b_1 + b_2 + d_1 + d_2 + c_1 + c_2 + h_1 + h_2]^{-1}.$$

The expected number of machines in the system is

$$L = \sum_{n=1}^{3}\sum_{s=1}^{2} n\, p_{n,s}$$

$$= \{d_1 + d_2 + c_1 + c_2 + 2(b_1 + b_2) + 3\,(a + 1)\}\, p_{3,2},$$

Machine efficiency (the fraction of total production time on all machines) is

$$U_m = 1 - \frac{L}{3}.$$

Average operator utilization

$$U_s = \sum_{n=0}^{2}\sum_{s=1}^{2} \frac{n\, p_{n,s}}{2} + \sum_{s=1}^{2} p_{3,s}$$

$$U_s = 1 - \left\{(h_1 + h_1) + \tfrac{1}{2}(d_1 + d_2 + c_1 + c_2)\right\} p_{3,2}\,.$$

6- special Cases

Some queuing systems can be obtained as special cases of this model:

(i) If $r = 1$ we get the model : M/M/2/ k /N, which was studied by Shawky [10] at $\alpha = 0$ and $\beta = 1$

(ii) If . $k = N \to \infty, r = 1, \pi_1 = 1$, and $\pi_2 = 0$ we obtain the queue M/M/2 which was studied by Singh[11].

(iii) If $k = N = m, \mu_1 = \mu_2$, $r = 1$ and $\pi_1 = \pi_2 = 1/2$,we get the homogeneous repairmen model : M/M/2/ m / m which was discussed by White et al. [13], Gross and Harris [7] and Bunday [2].

7-Conclusion

In this paper, the machine interference model: $H_r\ /M/2/k/N$ is studied with two heterogeneous repairmen . The recurrence relations that give all the probabilities in terms of $P_{k,r}$ are derived. We illustrated the method by a numerical example and deduced the expected number of units in the system and machine efficiency, also the average operator utilization are derived. Some special cases are obtained.

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