INVERTED GOMPERTZ DISTRIBUTION:

PROPERTIES AND ESTIMATION

By

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ABSTRACT

The inverted Gompertz distribution, and some of its properties are investigated. The distribution of the r^{th} order statistic of the inverted Gompertz distribution and some special cases are presented. Transformed distributions of this distribution are derived. Based on Type II censored samples, the estimation of the shape and scale parameters, reliability and hazard rate functions are obtained using the maximum likelihood method. The uniformly minimum variance unbiased estimate of a function of the shape parameter is obtained assuming that the scale parameter is known. Asymptotic variances and covariances of the maximum likelihood estimators are found. Confidence intervals of the parameters are also given. Via a Monte Carlo simulation study, the estimated risks, squared bias and variances of the estimates are computed from different sample sizes.

Keywords and phrases: *Inverted Gompertz distribution; reliability function; hazard rate function; Type II censored samples; maximum likelihood method; uniformly minimum variance unbiased estimate.*

1. INTRODUCTION

The Gompertz distribution plays an important role in modeling human mortality and fitting actuarial tables. Benjamin Gompertz introduced a new distribution named by Gompertz distribution [see AL-Hussaini, AL-Dayian and Adham (2000)]. It is used as a growth model, especially in epidemiological and biomedical studies [see Jaheen (2003b)]. Casey used the Gompertz distribution as a statistical model to fit tumor growth [see AL-Hussaini, AL-Dayian and Adham (2000)]. Also, Laird studied several types of tumors in mice, rats and rabbits and concluded that the growth of a transplanted, or primary, tumor is described well by the Gompertz distribution [see AL-Hussaini, AL-Dayian and Adham (2000)].

Many authors have contributed to the studies of statistical methodology and characterization of this distribution; for example, Sherman and Morrison (1950), Ahuja and Nash (1967), Garg, Raja Rao and Redmond (1970), Adham (1996), AL-Hussaini, AL-Dayian and Adham (2000), Adham and Walker (2001), Jaheen [(2003a), (2003b)], Wu, Hung and Tsai (2004), Hendi, Abu-Youssef and Alraddadi (2006) and AL-Khedhairi and EL-Gohary (2008).

The probability density function (pdf) of the random variable(X) which has a Gompertz distribution is defined as;

$$f(x) = \text{a}\ \exp\left[bx - \frac{\text{a}}{b}\left(\exp\,(bx) - 1\right) \right], \qquad x > 0, \quad (\text{a}, b > 0), \tag{1}$$

then, the distribution of $T = X^{-1}$ is referred as the inverse or inverted Gompertz (IG) distribution.

This paper consists of five sections. Section (1) is an introduction. In Section (2), some descriptive properties of the inverted Gompertz distribution are obtained. Also, some transformed distributions of the inverted Gompertz distribution are given. In Section (3), maximum likelihood estimation of the shape and scale parameters, reliability function (RF) and hazard rate function (HRF) of the inverted Gompertz distribution based on Type II censored samples are presented. In Section (4), a Monte Carlo simulation study is described. This study ends, in Section (5), by concluding remarks.

2. THE MAIN PROPERTIES OF THE INVERTED GOMPERTZ DISTRIBUTION

This section is devoted to the description of the IG distribution. Main properties of the IG distribution are obtained. Graphical description is presented. Moments and quantiles of the IG distribution are derived. Finally, the distribution of the r^{th} order statistic of the IG distribuiton is obtained.

Let T be a random variable distributed as IG distribution with shape parameter $\text{a} > 0$ and scale parameter $b > 0$, denoted by $T \sim \text{IG}(\text{a}, b)$, then

1. The pdf of the random variable T, using (1), is given by;

$$f(t) = \text{a}\ t^{-2} \exp\left[b t^{-1} - \frac{\text{a}}{b}\left(\exp\,(b t^{-1}) - 1\right) \right] \qquad ;\ t > 0, \quad (\text{a}, b > 0). \tag{2}$$

2. The cumulative distribution function (cdf) of the random variable T is given by;

$$F(t) = \exp\left[\frac{-a}{b}\left(\exp(b\,t^{-1}) - 1\right)\right] \qquad ; t > 0,\ (a, b > 0). \qquad (3)$$

3. The RF of the random variable T is given by;

$$R(t) = 1 - \exp\left[\frac{-a}{b}\left(\exp(b\,t^{-1}) - 1\right)\right] \qquad ; t > 0,\ (a, b > 0). \qquad (4)$$

4. The HRF of the random variable T is given by;

$$h(t) = \frac{a\ t^{-2} \exp\left[b\,t^{-1} - \frac{a}{b}\left(\exp(b\,t^{-1}) - 1\right)\right]}{1 - \exp\left[\frac{-a}{b}\left(\exp(b\,t^{-1}) - 1\right)\right]} \qquad ; t > 0,\ (a, b > 0). \qquad (5)$$

5. Graphical description

The curves of four IG(a, b) densities and the corresponding HRF's are plotted in figure (1).

Figure (1): Density and HRF's of IG (a, b) Distributions

(1) $a > b$　　　　(2) $a < b$

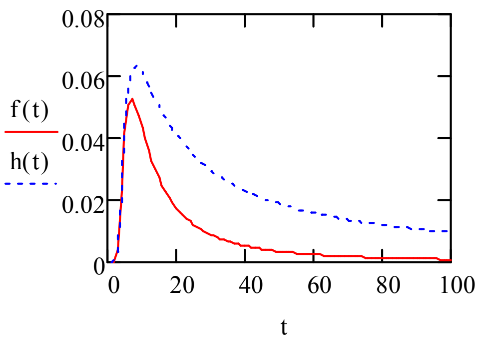

Fig. (1.a): a=10, b=4

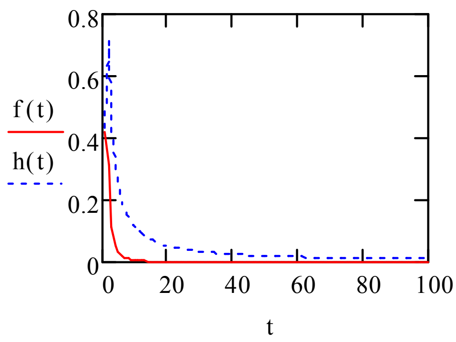

Fig. (1.b): a=0.5, b=3

(3) a = b

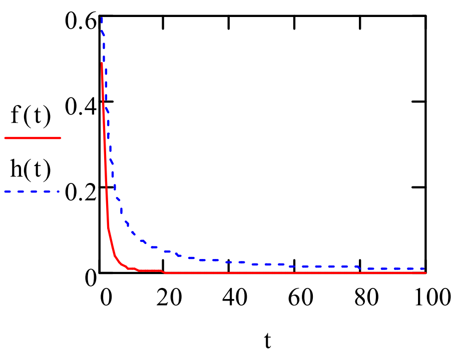

Fig. (1.c): a=1, b=1

(4) $b = 1$

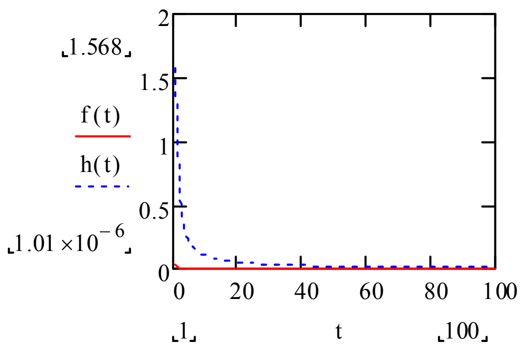

Fig. (1.d): a=0.01 , b=1

Figure (1) shows that:

- The density curve of the $\mathrm{IG}(\mathrm{a}, b)$ is positively skewed if $\mathrm{a} > 0$.
- When $\mathrm{a} > b$ the curves of the pdf and HRF are monotone increasing and then monotone decreasing.
- When $\mathrm{a} < b$ the curves of the pdf and HRF are monotone decreasing.
- When $\mathrm{a} = b = 1$ the curves of the pdf and HRF are monotone decreasing.

6. Moments of the inverted Gompertz distribution

The moments of the $\mathrm{IG}(\mathrm{a}, b)$ distribution do not exist, but the approximate mean and variance, using the method of statistical differentials [see Adham (1996)]., for the $\mathrm{IG}(\mathrm{a}, b)$ distribution can be written as follows:

$$E(T) \cong \left[\frac{1}{b}\ln\left(1+\frac{b}{\mathrm{a}}\right)\right]^{-3}\left\{\left[\frac{1}{b}\ln\left(1+\frac{b}{\mathrm{a}}\right)\right]^{2}+\frac{1}{2\mathrm{a}^{2}}\left(1+\frac{b}{\mathrm{a}}\right)^{-2}\left[2+\ln\left(1+\frac{b}{\mathrm{a}}\right)\right]\right\}, \tag{6}$$

and

$$V(T) \cong \frac{1}{\mathrm{a}^{2}}\left(1+\frac{b}{\mathrm{a}}\right)^{-2}\left[\frac{1}{b}\ln\left(1+\frac{b}{\mathrm{a}}\right)\right]^{-4}. \tag{7}$$

[For proof, see the Appendix].

7. The mode of the inverted Gompertz distribution

The mode D of the random variable T can be obtained by maximizing the pdf as followes:

$$0 = f'(D) = f(D)\ A(D), \tag{8}$$

where $f(D)$ is as given by (2), after replacing t by D, and

$$A(D) = D^{-2}\left[\mathrm{a}\ \exp\left(b\, D^{-1}\right) - b\ \right] - 2D^{-1}\ . \tag{9}$$

The solution of (6) is $A(D) = 0$ which gives the mode of an IG(a, b).

8. Quantiles of the inverted Gompertz distribution

The quantile of the IG(a, b) distribution is given by

$$t_q = \left[\frac{1}{b}\ln\left(1 - \frac{b}{\mathrm{a}}\ln q\right)\right]^{-1}, \qquad 0 < q < 1, \tag{10}$$

and the special cases may be obtained by using (10) such as first and third quartiles, when $q = 1/4$ and $q = 3/4$, respectively. Also, if $q = 1/2$, we obtain the median of T, which is given by

$$\text{median} \equiv t_{median} = \left[\frac{1}{b}\ln\left(1 + \frac{b}{\mathrm{a}}\ln(2)\right)\right]^{-1}. \tag{11}$$

9. The distribution of the r^{th} order statistic of the inverted Gompertz distribution

Let $T_1, \ldots, T_n$ be independent identically distributed random variables from an IG(a, b) distribution. Let $T_{(r)}$ denote the r^{th} order statistic of $T_1, \ldots T_n$. Then the pdf of $T_{(r)}$ can be written as a linear combination of IG(α_{rj}, b) density functions. That is,

$$f_{T_{(r)}}(t) = \sum_{j=0}^{n-r} \alpha_{rj}\, f(t \mid \mathrm{a}(j+r), b), \qquad t > 0, \tag{12}$$

where, for $j = 0, 1, \ldots, n-r$, $f(t \mid \mathrm{a}(j+r), b)$ is as given by (2) after replacing t and a by $t_{(r)}$ and $\mathrm{a}(j+r)$, respectively, and the combination factors α_{rj} are given by

$$\alpha_{rj} = \frac{(-1)^j r}{(j+r)}\binom{n}{r}\binom{n-r}{j}. \tag{13}$$

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Proof:

It is well known that the pdf of the r^{th} order statistic $T_{(r)}$ of a random sample of size *n* drawn from a population with pdf $f_T(t)$ and cdf $F_T(t)$ is given by

$$f_{T_{(r)}}(t) = r\binom{n}{r} f_T(t)\left[F_T(t)\right]^{r-1}\left[1-F_T(t)\right]^{n-r}, \quad t>0. \tag{14}$$

By expanding $\left[1-F_T(t)\right]^{n-r}$, using the binomial expansion, (14) can be written as

$$f_{T_{(r)}}(t) = r\binom{n}{r} f_T(t)\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\left[F_T(t)\right]^{r+j-1}, \quad t>0. \tag{15}$$

By substituting $f_T(t)$, $F_T(t)$ given by (2) and (3) in (15), after replacing t by $t_{(r)}$, we obtain

$$f_{T_{(r)}}(t) = r\binom{n}{r} \text{a}\, t^{-2}\exp(bt^{-1})\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\left(\exp\left[\frac{-\text{a}}{b}\left(\exp(bt^{-1})-1\right)\right]\right)^{r+j}$$

$$= \sum_{j=0}^{n-r}\alpha_{rj}\,\text{a}(j+r)\,t^{-2}\exp\left[bt^{-1}-\frac{\text{a}(j+r)}{b}\left(\exp(bt^{-1})-1\right)\right] = \sum_{j=0}^{n-r}\alpha_{rj}\, f(t\,|\,\text{a}(j+r),b),$$

where

$$\alpha_{rj} = \frac{(-1)^j r}{(j+r)}\binom{n}{r}\binom{n-r}{j}.$$

Special cases

(i) If, in (12), $r=1$, we obtain the pdf of the first order statistic, $T_{(1)} = \min\limits_{1\le \text{a}\le n}\{T_\text{a}\}$, which is given by

$$f_{T_{(1)}}(t) = \sum_{j=0}^{n-1}\alpha_{1j}\, f_j(t\,|\,\text{a}(j+1),b), \qquad t>0, \tag{16}$$

where

$$\alpha_{1j} = (-1)^j\binom{n}{j+1}, \quad j=0,1,\ldots,n-1. \tag{17}$$

and $f_{T_{(1)}}(t\,|\,\text{a}(j+1),b)$ is the pdf of the IG(a,b), given by (2) after replacing a by a(j+1), $j=0,1,\ldots,n-1$. Therefore, the pdf of the minimum of a random sample drawn from an

$\mathrm{IG}(\mathrm{a},b)$ population is a linear combination of $\mathrm{IG}\left(\mathrm{a}(j+1),b\right)$ density functions, and combination factors α_{1j} given by (17), where $j=0,1,\ldots,n-1$.

(ii) If the sample size is odd, let $n = 2m + 1$ and in (12), $r = m + 1$, then the pdf of the median is given by

$$f_{T_{(m+1)}}(t) = \sum_{j=0}^{m} \alpha_{(m+1)j}\, f_j(t \mid \mathrm{a}(j+m+1),b), \qquad t>0, \tag{18}$$

where

$$\alpha_{(m+1)j} = \frac{(-1)^j (m+1)}{(j+m+1)} \binom{2m+1}{m+1} \binom{m}{j}, \tag{19}$$

and $f(t \mid \mathrm{a}(j+m+1),b)$, is the pdf of the $\mathrm{IG}(\mathrm{a},b)$, given by (2) after replacing a by $\mathrm{a}(j+m+1)$, $j=0,1,\ldots,m$. Therefore, the pdf of the median of a random sample drawn from an $\mathrm{IG}(\mathrm{a},b)$ population is a linear combination of $\mathrm{IG}\left(\mathrm{a}(j+m+1),b\right)$ density functions, and combination factors $\alpha_{(m+1)j}$ given by (19), where $j=0,1,\ldots,m$.

(iii) If, in (12), $r = n$, we obtain the pdf of the last order statistic, $T_{(n)} = \max\limits_{1\le \mathrm{a} \le n}\{T_{\mathrm{a}}\}$, which is given by

$$f_{T_{(n)}}(t) = \sum_{j=0}^{n-1} f(t \mid \mathrm{a}(n),b), \qquad t>0, \tag{20}$$

where $f_{T_{(n)}}(t \mid \mathrm{a}(n),b)$ is the pdf of the $\mathrm{IG}(\mathrm{a},b)$ given by (2) after replacing a by $\mathrm{a}(n)$. Therefore, the pdf of the maximum of a random sample drawn from an $\mathrm{IG}(\mathrm{a},b)$ population is the same distribution but with parameters $\mathrm{a}(n)$ and b as shown in (20).

10. Transformations applied to the inverted Gompertz distribution and the resulting distributions

It can be shown that the $\mathrm{IG}(\mathrm{a},b)$ distribution is related through variable transformations to a wide range of well known distributions such as Pareto Type I, Weibull (exponential, Rayleigh), gamma (chi-square), left-truncated exponential, Burr Type XII (Lomax, beta Type II, Pareto Type II, Pareto Type III, Pareto Type IV, log logistic, F), Generalized logistic Type I (logistic, standard logistic, Generalized logistic Type II, Burr Type II), Burr Type III, Burr Type IV, Burr Type X, extreme value, power-function, compound Gompertz and generalized uniform (beta Type I) distributions. Table (1) summarizes the transformations from $\mathrm{IG}(\mathrm{a},b)$ to other distributions. The proof, in each case, is straightforward.

Table (1)

Summary of Transformations Applied to the Inverted Gompertz and Resulting Distributions

Transformation	Distribution	Pdf	Range
$k \exp\left[\frac{1}{b}\left(\exp(bT^{-1})-1\right)\right]$	Pareto Type I (a,k)	$\mathrm{a}\ k^{\mathrm{a}}\ s_1^{-(\mathrm{a}+1)}$	$s_1 > k$, $(\mathrm{a},k>0)$
$\left[\frac{1}{b}\left(\exp(bT^{-1})-1\right)\right]^{\frac{1}{c}}$ *special cases*	Weibull (a,c)	$\mathrm{a}\ c\ s_2^{c-1}\ \exp(-\mathrm{a}\, s_2^{c})$	$s_2>0$, $(\mathrm{a},c>0)$
(i) $c=1$	exponential (a)		
(ii) $c=2$	Rayleigh (a)		
$\sum_{i=1}^{n}\left[\frac{1}{b}\left(\exp(bT^{-1})-1\right)\right]$ *special case*	gamma (n, a)	$\frac{\mathrm{a}^n}{\Gamma(n)}\ s_3^{n-1}\ \exp(-\mathrm{a}\, s_3)$	$s_3>0$, $(n,\mathrm{a}>0)$
(i) $\mathrm{a}=\frac{1}{2},\ n=\frac{m}{2}$	$\chi^2(m)$		
$\frac{\alpha}{b}\left[\exp(bT^{-1})-1\right]+c$	left-truncated exponential (a,α,c)	$\frac{\mathrm{a}}{\alpha}\exp\left[-\mathrm{a}\left(\frac{s_4-c}{\alpha}\right)\right]$	$s_4>c$, $(c<\alpha,\ \alpha,\mathrm{a}>0)$
$\left\{\lambda\left[\exp\left[\frac{1}{b}\left(\exp(bT^{-1})-1\right)\right]-1\right]\right\}^{\frac{1}{c}}+\xi$ *specialcases*	Burr Type XII $(c,\mathrm{a},\xi,\lambda)$	$\frac{c\,\mathrm{a}}{\lambda}(s_5-\xi)^{c-1}\left[1+\frac{(s_5-\xi)^c}{\lambda}\right]^{-(\mathrm{a}+1)}$	$s_5>\xi$, $(c,\mathrm{a},\xi,\lambda>0)$
(i) $\xi=0, c=1$	Lomax (λ,a) or compound gamma		
(ii) $\xi=0, c=1, \lambda=1$	beta Type II (1, a)		
(iii) $c=1$	Pareto Type II (ξ,λ,a)		
(iv) $\mathrm{a}=1, c=\gamma^{-1}, \lambda=\sigma^{\gamma^{-1}}$	Pareto Type III (ξ,σ,γ)		

Table (1) (continued)

Transformation	Distribution	Pdf	Range
(v) $c=\gamma^{-1},\lambda=\sigma^{\gamma^{-1}}$	Pareto Type IV $(\xi,\sigma,\gamma,\mathrm{a})$		
(vi) $\xi=0,\mathrm{a}=1,\lambda=\beta^{c}$	log logistic (c,β)		
(vii) $\xi=0,c=1,\lambda=\mathrm{a}$	F-distribution (2,2a)		
$\ln\left[\exp\left[\frac{1}{b}\left(\exp(bT^{-1})-1\right)\right]-1\right]^{-c}+\alpha$ *special cases*	Generalized logistic Type I (α,c,a)	$\dfrac{\frac{\mathrm{a}}{c}\exp\left[-\left(\frac{s_6-\alpha}{c}\right)\right]}{\left[1+\exp\left[-\left(\frac{s_6-\alpha}{c}\right)\right]\right]^{\mathrm{a}+1}}$	$-\infty<s_6<\infty$, $(-\infty<\alpha<\infty$, $c,\mathrm{a}>0)$
(i) $\mathrm{a}=1$	logistic (α,c)		
(ii) $\alpha=0,c=1,\mathrm{a}=1$	standard logistic		
(iii) $\alpha=0,c=\beta^{-1}$	generalized logistic Type II (β,a)		
(iv) $\alpha=0,c=1$	Burr Type II (a)		
$\left[\exp\left[\frac{1}{b}\left(\exp(bT^{-1})-1\right)\right]-1\right]^{\frac{-1}{c}}$	Burr Type III (c,a)	$c\,\mathrm{a}\,s_7^{-c-1}\left(1+s_7^{-c}\right)^{-\mathrm{a}-1}$	$s_7>0$, $(c,\mathrm{a}>0)$
$c\left[1+\left(\exp\left[\frac{1}{b}\left(\exp(bT^{-1})-1\right)\right]-1\right)^{c}\right]^{-1}$	Burr Type IV (c,a)	$\dfrac{\frac{\mathrm{a}}{s_8^2}\left\{(c-s_8)/s_8\right\}^{\frac{1}{c}-1}}{\left[1+\left\{(c-s_8)/s_8\right\}^{\frac{1}{c}}\right]^{\mathrm{a}+1}}$	$0<s_8<c$, $(c,\mathrm{a}>0)$
$\left[\ln\left(1-\exp\left[\frac{-1}{b}\left(\exp(bT^{-1})-1\right)\right]\right)^{-1}\right]^{\frac{1}{2}}$	Burr Type X (a)	$2\mathrm{a}s_9\exp(-s_9^2)\left[1-\exp(-s_9^2)\right]^{\mathrm{a}-1}$	$s_9>0$, $(\mathrm{a}>0)$
$\ln\left[\frac{1}{b}\left(\exp(bT^{-1})-1\right)\right]^{\frac{-1}{\alpha}}$	extreme value (a,α)	$\mathrm{a}\alpha\exp\left[-\alpha s_{10}-\mathrm{a}\exp(-\alpha s_{10})\right]$	$-\infty<s_{10}<\infty$, $(\mathrm{a},\alpha>0)$
$\exp\left[\frac{-1}{b}\left(\exp(bT^{-1})-1\right)\right]$	power-function(a,1)	$\mathrm{a}\,s_{11}^{\mathrm{a}-1}$	$0<s_{11}<1$ $(\mathrm{a}>0)$
$\ln\left[c\alpha\left(\exp\left[\frac{1}{b}\left(\exp(bT^{-1})-1\right)\right]-1\right)+1\right]^{\frac{1}{c}}$	compound Gompertz (c,a,α)	$\frac{\mathrm{a}}{\alpha}\exp(c\,s_{12})\left[1+\frac{\left[\exp(c\,s_{12})-1\right]}{c\alpha}\right]^{-(\mathrm{a}+1)}$	$s_{12}>0$, $(c,\mathrm{a},\alpha>0)$
$\left[\dfrac{1-\exp\left[\frac{-1}{b}\left(\exp(bT^{-1})-1\right)\right]}{\alpha}\right]^{\frac{1}{c}}+\beta$ *special case*	generalized uniform $(c,\mathrm{a},\alpha,\beta)$	$\dfrac{\mathrm{a}\,c\,\alpha\,(s_{13}-\beta)^{c-1}}{\left[1-\alpha\,(s_{13}-\beta)^{c}\right]^{-\mathrm{a}+1}}$	$\beta\le s_{13}\le\beta+\alpha^{\frac{-1}{c}}$, $(c,\mathrm{a},\alpha,\beta>0)$
(i) $\alpha=1,c=1,\beta=0$	beta Type I (1,a)		

3. ESTIMATION OF THE PARAMETERS, RELIABILITY AND HAZARD RATE FUNCTIONS BASED ON TYPE II CENSORED SAMPLES

In this section, we are concerned with the estimation of the vector of two parameters, $\underline{\theta} = (\mathrm{a}, b)'$, RF and HRF of the inverted Gompertz based on Type II censored samples. The maximum likelihood (ML) method is used to estimate the vector $\underline{\theta}$, RF and HRF. Furthermore, the uniformly minimum variance unbiased estimator (UMVUE) of the function of shape parameter, a, is obtained when the scale parameter, *b*, is known.

Suppose that $T_{(1)} \le T_{(2)} \le \cdots \le T_{(r)}$ is a censored sample of size *r* obtained from a life-test on *n* items (Type II censored sample) whose lifetimes have an $\mathrm{IG}(\mathrm{a}, b)$ distribution. The likelihood function based on Type II censored sample is given by

$$L(\mathrm{a}, b \mid \underline{t}) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(t_i; \mathrm{a}, b)\right]\left[R(t_{(r)}; \mathrm{a}, b)\right]^{n-r}, \tag{21}$$

where, $R(t_{(r)}; \mathrm{a}, b)$ is given by (4). When $r = n, L(\mathrm{a}, b \mid \underline{t})$, reduces to the likelihood function for complete samples.

The natural logarithm of the likelihood function, (21), is given by

$$\ell \equiv \ln L(\mathrm{a}, b \mid \underline{t}) = \ln\left(\frac{n!}{(n-r)!}\right) + \sum_{i=1}^{r} \ln\left[f(t_i; \mathrm{a}, b)\right] + (n-r)\ln\left[R(t_{(r)}; \mathrm{a}, b)\right]. \tag{22}$$

And the substitution of (2) and (4) into (22) yields

$$\ell \equiv \ln L(\mathrm{a}, b \mid \underline{t}) \propto r \ln \mathrm{a} - 2\sum_{i=1}^{r} \ln t_i + b\sum_{i=1}^{r} t_i^{-1} - \frac{\mathrm{a}}{b}\sum_{i=1}^{r} A(t_i) + (n-r)\ln\left[1 - \exp\left(\frac{-\mathrm{a}}{b}\,(A(t_r))\right)\right], \tag{23}$$

where

$$A(t_i) = \left(\exp\,(b t_i^{-1}) - 1\right) \text{ and } A(t_r) = \left(\exp\,(b t_r^{-1}) - 1\right). \tag{24}$$

3.1 MLE's of the Parameters of Inverted Gompertz Distribution

To obtain the MLE's of the parameters, a and *b*, differentiate ℓ in (23) with respect to a and *b* and setting to zero, we obtain

$$\frac{\partial \ell}{\partial \mathrm{a}} = \frac{r}{\hat{\mathrm{a}}} - \frac{1}{\hat{b}}\sum_{i=1}^{r} \hat{A}(t_i) + \frac{(n-r)\ \hat{A}(t_r)\ \exp\left(\frac{-\hat{\mathrm{a}}}{\hat{b}}\left(\hat{A}(t_r)\right)\right)}{\hat{b}\left[1 - \exp\left(\frac{-\hat{\mathrm{a}}}{\hat{b}}\left(\hat{A}(t_r)\right)\right)\right]} = 0, \tag{25}$$

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$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^{r} t_i^{-1} - \frac{\hat{a}}{\hat{b}^2}\left[\sum_{i=1}^{r} \exp(\hat{b}t_i^{-1})\left[\hat{b}t_i^{-1}-1\right] + r\right] + (n-r)\frac{\hat{a}}{\hat{b}^2}\exp\left(\frac{-\hat{a}}{\hat{b}}\left(\hat{A}(t_r)\right)\right) * \frac{\left[\hat{b}t_r^{-1}\exp(\hat{b}t_r^{-1}) - \hat{A}(t_r)\right]}{\left[1-\exp\left(\frac{-\hat{a}}{\hat{b}}\left(\hat{A}(t_r)\right)\right)\right]} = 0, \tag{26}$$

where $\hat{A}(t_i)$ and $\hat{A}(t_r)$ are as given by (24) after replacing b by $\hat{b}$.

The two nonlinear likelihood equations (25) and (26) can be solved by using Newton-Raphson iteration scheme, to yield the MLE's $(\hat{a},\hat{b})$ of a and b.

Remark (1):

It may be observed that the complete sample case is a special case from Type II censored sample and in this case, we notice the following:

- The two nonlinear likelihood equations (25) and (26) can be reduced to one equation as a function of b only, after obtaining $\hat{a}$ from (25) and substituting the MLE of a in (26) and solving it numerically.
- Suppose that $T_1, T_2, \ldots, T_n$ is a random sample of size n drawn from an $\mathrm{IG}(a, b)$ population with pdf given by (2), where b is known and a is unknown. If $\Psi = a^{-1}$, then the UMVUE of Ψ is $\frac{1}{nb}\sum_{i=1}^{n} A(t_i)$, that is, $\hat{\Psi} = \sum_{i=1}^{n}\frac{1}{nb}A(t_i)$, where $A(t_i) = \left(\exp(bt_i^{-1}) - 1\right)$.

3.2 MLE's of RF and HRF

The MLE's of the RF, $R(t)$, and HRF, $\mathrm{h}(t)$, are obtained by replacing the parameters a and b in (4) and (5) by their MLE's. Hence, for a given value of t, the MLE's of $R(t)$ and $\mathrm{h}(t)$ are given, respectively, by

$$\hat{R}(t_0) = 1 - \exp\left[\frac{-\hat{a}}{\hat{b}}\left(\exp(\hat{b}t_0^{-1}) - 1\right)\right], \quad t_0 > 0, \ (\hat{a}, \hat{b} > 0), \tag{27}$$

$$\hat{\mathrm{h}}(t_0) = \frac{\hat{a}\, t_0^{-2}\exp\left[\hat{b}t_0^{-1} - \frac{\hat{a}}{\hat{b}}\left(\exp(\hat{b}t_0^{-1}) - 1\right)\right]}{1 - \exp\left[\frac{-\hat{a}}{\hat{b}}\left(\exp(\hat{b}t_0^{-1}) - 1\right)\right]}, \quad t_0 > 0, \ (\hat{a}, \hat{b} > 0), \tag{28}$$

where $\hat{a}$ and $\hat{b}$ are the MLE's of a and b respectively.

3.3 Asymptotic Variances and Covariances of MLE's

The asymptotic variances and covariances of MLE's are given by the elements of the inverse of the Fisher information matrix $I_{ij}(\underline{\theta}) \equiv E\left\{-\partial^2 \ell / \partial\theta_i \, \partial\theta_j\right\}$ where $i, j = 1, 2$ and $\underline{\theta} = (\mathrm{a}, b)'$. Unfortunately, the exact mathematical expressions for the above expectations are very difficult to obtain. Therefore, we give below the observed Fisher information matrix $\hat{I}_{ij}(\underline{\theta}) \equiv \left\{-\partial^2 \ell / \partial\theta_i \, \partial\theta_j\right\}$, which is obtained by dropping the expectation on operator *E*. The approximate asymptotic variances-covariances matrix $\hat{V}$ for the MLE's is obtained by inverting the observed information matrix,

$$\left[\hat{V}\right] = \left[\hat{I}_{ij}(\underline{\theta})\right]^{-1}. \tag{29}$$

Differentiating (25) and (26) with respect to a and *b* we have:

$$\frac{\partial^2 \ell}{\partial \mathrm{a}^2} = \frac{-r}{\mathrm{a}^2} - \frac{(n-r)}{b^2} A^2(t_r)\, \exp\left(\frac{-\mathrm{a}}{b}\big(A(t_r)\big)\right)\left[1 - \exp\left(\frac{-\mathrm{a}}{b}\big(A(t_r)\big)\right)\right]^{-2}, \tag{30}$$

$$\frac{\partial^2 \ell}{\partial b^2} = \frac{-\mathrm{a}}{b} \sum_{i=1}^{r} t_i^{-2} \exp(b t_i^{-1}) + \frac{2\mathrm{a}}{b^3}\left[\sum_{i=1}^{r} \exp(b t_i^{-1})\left[b t_i^{-1} - 1\right] + r\right] + (n-r)\,\frac{\mathrm{a}}{b^4}\,\exp\left(\frac{-\mathrm{a}}{b}\big(A(t_r)\big)\right) * \frac{\left[1-\exp\left(\frac{-\mathrm{a}}{b}\big(A(t_r)\big)\right)\right]\left\{b^3 t_r^{-2}\exp(b t_r^{-1}) + \left[\exp(b t_r^{-1})\left(b t_r^{-1}-1\right)+1\right]\left[\mathrm{a}\exp(b t_r^{-1})\left(1-b t_r^{-1}\right) - \mathrm{a} - 2b\right]\right\} - \frac{\mathrm{a}^2}{b^4}\exp\left(\frac{-2\mathrm{a}}{b}\big(A(t_r)\big)\right)\left[\exp(b t_r^{-1})\left(b t_r^{-1}-1\right)+1\right]^2}{\left[1-\exp\left(\frac{-\mathrm{a}}{b}\big(A(t_r)\big)\right)\right]^2} \tag{31}$$

$$\frac{\partial^2 \ell}{\partial b \, \partial \mathrm{a}} = \frac{-1}{b^2}\left[\sum_{i=1}^{r} \exp(b t_i^{-1})\left[b t_i^{-1} - 1\right] + r\right] + \frac{(n-r)}{b^2}\left[\exp(b t_r^{-1})\left(b t_r^{-1} - 1\right) + 1\right] \frac{\exp\left(\frac{-\mathrm{a}}{b}\big(A(t_r)\big)\right)\left\{1 - \frac{\mathrm{a}}{b}\big(A(t_r)\big) - \exp\left(\frac{-\mathrm{a}}{b}\big(A(t_r)\big)\right)\right\}}{\left[1-\exp\left(\frac{-\mathrm{a}}{b}\big(A(t_r)\big)\right)\right]^2}, \tag{32}$$

where $A(t_r)$ is as given by (24).

3.4 Confidence Intervals of a

Assuming the shape parameter is the only unknown parameter and $r = n$, then the confidence interval (C.I.) for a is given by

$$P\left[\frac{\chi^2_{\frac{\alpha}{2}}}{2y} < \mathrm{a} < \frac{\chi^2_{1-\frac{\alpha}{2}}}{2y}\right] = 1 - \alpha \ ,$$

where, $Y = \sum_{i=1}^{n} Y_{1i} \sim gamma(n, \mathrm{a})$ and $Y_1 = \dfrac{\left(\exp\,(bT^{-1}) - 1\right)}{b} \sim$ exponential (a), [see Table (1) the third transformation], χ^2 is a tabulated value of chi-square at *2n* degrees of freedom. However, approximate confidence intervals for a and b can be developed by invoking the asymptotic normality of the MLE's. Hence $100(1-\alpha)\%$ central confidence intervals for a and b are given, respectively, by

$$\hat{a} \pm z_{\alpha/2}\sqrt{Var(\hat{a})}\ , \qquad (33)$$

$$\hat{b} \pm z_{\alpha/2}\sqrt{Var(\hat{b})}\ , \qquad (34)$$

where $z_{\alpha/2}$ is a standard normal variate.

4. MONTE CARLO SIMULATION

Some numerical results, based on the MLE's, are obtained according to the following steps:

1. Given a and b, generate random samples of size n (n = 40, 60 and 100) from an $\mathrm{IG}(a,b)$, by observing that if U is uniform (0, 1), then $T=\left[\frac{1}{b}\ln\left(1-\frac{b}{a}\ln U\right)\right]^{-1}$ is $\mathrm{IG}(a,b)$.
2. Order the sample obtained in step1.
3. Obtain the MLE's of the parameters a and b by solving the two nonlinear equations (25) and (26) simultaneously, using Newton-Raphson iteration scheme.
4. Compute the MLE's of the RF and HRF at time t_0, using equations (27) and (28), respectively, after replacing a and b by their corresponding MLE's $\hat{a}$ and $\hat{b}$.
5. Repeat the above steps m times, where m = 1000 for n = 40, 60 and 100. The computations are carried out for censoring percentage of 80%, 90%, 95% and 100% (complete sample case) for each sample size.
6. The estimated risks (ER's), squared biases ($Bias^2$) and variances (Var) of the estimators are computed by averaging over the m repetitions.

   All simulation results presented here are obtained via the MathCad-14 mathematical package. The computational results are displayed in Tables (2), (3) and (4).

Table (2) presents the estimated risks, squared biases and variances of the MLE's of the two unknown parameters a and b. The "actual" population values are $a = 3.0, b = 3.0, R(t) = 0.9017379041, h(t) = 0.1736602066$, where $t_0 = 2.5$.

Table (3) presents the estimated risks, squared biases and variances of the MLE's of the two unknown parameters a and b. The "actual" population values are $a = 2.5, b = 3.5, R(t) = 0.6318248999, h(t) = 0.218416465$, where $t_0 = 4.0$.

Table (4) presents the estimated risks, squared biases and variances of the MLE's of the two unknown parameters a and b. The "actual" population values are $a = 0.8, b = 1.0, R(t) = 0.2712976578, h(t) = 0.3332087719$, where $t_0 = 3.0$.

5. CONCLUDING REMARKS

It may be observed, from Tables (2), (3) and (4), that

1. All the results based on censored samples of size r, can be specialized to the complete sample case by taking $r = n$.
2. The estimates $\hat{a}$ and $\hat{b}$, are better when $0 < a \le 1$ than when $a > 1$.
3. If the sample size, n, increases and the censoring ratio is kept fixed, the estimated risks (ER's) decrease and the estimates improved.
4. By increasing the censoring size, r, that is, by using more sample units the ER's and variances of the estimates decrease and the efficiency of the estimates increase as a result of decreasing variances.
5. The estimates $\hat{R}(t)$ and $\hat{h}(t)$ are almost asymptotically unbiased.
6. The covariances between the estimates $\hat{a}$ and $\hat{b}$ are negative in all cases.

Table (2)

Estimated Risks, Bias 2 and Variances of the MLE's of the shape and scale parameters, a and *b*, RF and HRF of the inverted Gompertz for Different Sample Sizes *n*, Censoring Sizes *r* and Repetitions *m*

(a = 3, *b* = 3, R(2.5) = 0.9017379041, h(2.5) = 0.1736602066)

n	r	$ER(\hat{a})$ $Bias^2(\hat{a})$ $Var(\hat{a})$	$ER(\hat{b})$ $Bias^2(\hat{b})$ $Var(\hat{b})$	$\mathrm{Cov}(\hat{a},\hat{b})$	$ER(\hat{R}(t_0))$ $Bias^2(\hat{R}(t_0))$ $Var(\hat{R}(t_0))$	$ER(\hat{h}(t_0))$ $Bias^2(\hat{h}(t_0))$ $Var(\hat{h}(t_0))$
40	32	0.7488454030 0.0001924052 0.7486529978	1.7442730131 0.1296665834 1.6146064297	-0.8730	0.0014635233 0.0000312930 0.0014322303	0.0027088308 0.0000123884 0.0026964423
	36	0.6624657841 0.0022829822 0.6601828018	1.6761982818 0.1482864401 1.5279118417	-0.7865	0.0014220116 0.0000276557 0.0013943559	0.0025547597 0.0000066782 0.0025480814
	38	0.6617548417 0.0021520811 0.6596027606	1.6718205878 0.1464401071 1.5253804807	-0.7849	0.0014218905 0.0000276347 0.0013942558	0.0025477492 0.0000070715 0.0025406777
	40	0.6467216096 0.0074529168 0.6392686929	1.5987579655 0.1783449851 1.4204129804	-0.7311	0.0014111554 0.0000196201 0.0013915353	0.0026183906 0.0000001142 0.0026182764
60	48	0.4878112385 0.0000352682 0.4877759702	1.1191805247 0.0578477053 1.0613328194	-0.5792	0.0009976174 0.0000186182 0.0009789992	0.0017506148 0.0000079501 0.0017426646
	54	0.4813309870 0.0000220848 0.4813089022	1.0948674987 0.055884868 1.0389826307	-0.5676	0.0009973140 0.0000183394 0.0009789746	0.0017305684 0.0000083616 0.0017222068
	57	0.4809455718 0.0000180431 0.4809275287	1.094269928 0.0555697645 1.0387001634	-0.5673	0.0009974443 0.0000183270 0.0009791173	0.0017314240 0.0000084290 0.0017229950
	60	0.4393910638 0.0004866889 0.4389043749	0.9620426338 0.0521046324 0.9099380014	-0.4921	0.0009516071 0.0000099313 0.0009416757	0.0015755224 0.0000024449 0.0015730775
100	80	0.2806393533 0.0019833314 0.2786560218	0.62093056 0.0341553407 0.5867752194	-0.3253	0.0006107013 0.0000037469 0.0006069544	0.0009840728 0.0000001847 0.0009838881
	90	0.2781131550 0.0018293863 0.2762837687	0.6105992556 0.0329311384 0.5776681172	-0.3206	0.0006102371 0.0000036918 0.0006065452	0.0009798267 0.0000001321 0.0009796946
	95	0.2818751844 0.0000130300 0.2818621544	0.5691646298 0.0154710886 0.5536935412	-0.3182	0.0005775943 0.0000056351 0.0005719592	0.0009746555 0.0000021825 0.0009724730
	100	0.2613203647 0.0008572109 0.2604631538	0.5621608389 0.0233302311 0.5388306078	-0.3035	0.0005582392 0.0000029238 0.0005553155	0.0009193430 0.0000000228 0.0009193202

Table (3)

Estimated Risks, Bias [2] and Variances of the MLE's of the shape and scale parameters, a and *b*, RF and HRF of the inverted Gompertz for Different Sample Sizes *n*, Censoring Sizes *r* and Repetitions *m*

(a = 2.5, *b* = 3.5, R(4) = 0.6318248999, h(4) = 0.218416465)

n	r	$ER(\hat{a})$ $Bias^2(\hat{a})$ $Var(\hat{a})$	$ER(\hat{b})$ $Bias^2(\hat{b})$ $Var(\hat{b})$	$\mathrm{Cov}(\hat{a},\hat{b})$	$ER(\hat{R}(t_0))$ $Bias^2(\hat{R}(t_0))$ $Var(\hat{R}(t_0))$	$ER(\hat{h}(t_0))$ $Bias^2(\hat{h}(t_0))$ $Var(\hat{h}(t_0))$
40	32	0.5320075257 0.0080611627 0.5239463630	1.6850831841 0.2103710777 1.4747121064	-0.7136	0.0045792848 0.0000196856 0.0045595992	0.0022840307 0.0001814260 0.0021026047
	36	0.4984301881 0.0012412202 0.4971889679	1.6124017567 0.1308380438 1.4815637129	-0.6870	0.0042552955 0.0000000255 0.0042552700	0.0020471292 0.0000900139 0.0019571153
	38	0.4976876979 0.0011453021 0.4965423958	1.6065856111 0.1287885233 1.4777970878	-0.6851	0.0042392658 0.0000000024 0.0042392634	0.0020329606 0.0000878234 0.0019451372
	40	0.4833616637 0.0048766643 0.4784849994	1.5377057399 0.1604765582 1.3772291817	-0.6375	0.0043062012 0.0000085975 0.0042976037	0.0019912067 0.0001265839 0.0018646228
60	48	0.3400788549 0.0022173859 0.3378614691	1.0398256983 0.0777524864 0.962073212	-0.4753	0.0027656856 0.0000033827 0.0027623029	0.0013460719 0.0000634299 0.0012826420
	54	0.3350006925 0.0017811603 0.3332195322	1.0105373441 0.0720489794 0.9384883648	-0.4646	0.0027283084 0.0000021251 0.0027261833	0.0013033766 0.0000569312 0.0012464454
	57	0.3293047917 0.0000583791 0.3292464126	0.9921804430 0.0416976711 0.9504827719	-0.4469	0.0027139613 0.0000005670 0.0027133943	0.0011940492 0.0000257155 0.0011683337
	60	0.3199839176 0.0009410660 0.3190428516	0.9213917112 0.0517132341 0.8696784771	-0.4241	0.0028172425 0.0000023443 0.0028148981	0.0012272437 0.0000414947 0.0011857490
100	80	0.2018884063 0.0002114496 0.2016769566	0.5248747283 0.0058687117 0.5190060167	-0.2671	0.0016441070 0.0000004882 0.0016436188	0.0006999805 0.0000037467 0.0006962338
	90	0.2016107332 0.0002085049 0.2014022283	0.5200420764 0.0058176686 0.5142244078	-0.2653	0.0016397200 0.0000004815 0.0016392385	0.0006928801 0.0000036935 0.0006891865
	95	0.2006612523 0.0001447931 0.2005164591	0.5443337011 0.0143443107 0.5299893904	-0.2669	0.0016682845 0.0000008728 0.0016674117	0.0007088750 0.0000122582 0.0006966168
	100	0.2004937387 0.0001347341 0.2003590047	0.5427332798 0.0141490524 0.5285842274	-0.2664	0.0016677261 0.0000008226 0.0016669035	0.0007071233 0.0000120443 0.0006950790

Table (4)

Estimated Risks, Bias ^{2} and Variances of the MLE's of the shape and scale parameters, a and *b*, RF and HRF of the inverted Gompertz for Different Sample Sizes *n*, Censoring Sizes *r* and Repetitions *m*

(a = 0.8, *b* = 1, R(3) = 0.2712976578, h(3) = 0.3332087719)

n	r	$ER(\hat{a})$ $Bias^2(\hat{a})$ $Var(\hat{a})$	$ER(\hat{b})$ $Bias^2(\hat{b})$ $Var(\hat{b})$	$\mathrm{Cov}(\hat{a},\hat{b})$	$ER(\hat{R}(t_0))$ $Bias^2(\hat{R}(t_0))$ $Var(\hat{R}(t_0))$	$ER(\hat{h}(t_0))$ $Bias^2(\hat{h}(t_0))$ $Var(\hat{h}(t_0))$
40	32	0.0540344821 0.0001145260 0.0539199561	0.1410219151 0.0130621170 0.1279597981	-0.0659	0.0031134041 0.0000117961 0.003101608	0.0009708845 0.0000548372 0.0009160473
	36	0.0533715092 0.0000773264 0.0532941829	0.1386749259 0.0122225558 0.1264523701	-0.0650	0.0030646121 0.0000086983 0.0030559138	0.0009505313 0.0000501110 0.0009004203
	38	0.0534332759 0.0000747800 0.0533584959	0.1382142105 0.0121618645 0.1260523460	-0.0649	0.0030686723 0.0000085237 0.0030601486	0.0009489165 0.0000497788 0.0008991377
	40	0.0517310823 0.0001108668 0.0516202155	0.1372866437 0.0099800627 0.1273065809	-0.0639	0.0029455515 0.0000141485 0.0029314030	0.0009369049 0.0000451278 0.0008917771
60	48	0.0337366571 0.0000943719 0.0336422852	0.0915579298 0.0071075787 0.0844503511	-0.0431	0.0018974741 0.0000064512 0.0018910229	0.0006175422 0.0000294383 0.0005881039
	54	0.0328921033 0.0000760136 0.0328160896	0.0891344842 0.0067220722 0.0067220720	-0.0418	0.0018533316 0.0000050927 0.0018482389	0.0005990692 0.0000272706 0.0005717986
	57	0.0328911099 0.0000750923 0.0328160175	0.0891241589 0.0067093701 0.0824147889	-0.0419	0.0018538744 0.0000050405 0.0018488339	0.0005991271 0.0000271995 0.0005719276
	60	0.0328697869 0.0000716081 0.0327981787	0.0887803349 0.0066437188 0.0821366161	-0.0418	0.0018519223 0.0000048247 0.0018470976	0.0005970681 0.0000268394 0.0005702287
100	80	0.0218334814 0.0000034678 0.0218300137	0.0505546472 0.0017544691 0.0488001781	-0.0270	0.0012456793 0.0000007287 0.0012449506	0.0003666754 0.0000068993 0.0003597761
	90	0.0214276161 0.0000045858 0.0214230303	0.0499484191 0.0017642010 0.0481842181	-0.0265	0.0012250690 0.0000007974 0.0012242716	0.0003608020 0.0000069619 0.0003538401
	95	0.0214224288 0.0000033003 0.0214191285	0.0497088953 0.0017179711 0.0479909242	-0.0265	0.0012244566 0.0000006697 0.0012237869	0.0003594730 0.0000067092 0.0003527638
	100	0.0213915303 0.0000033012 0.0213882291	0.0496819192 0.0017159876 0.0479659316	-0.0264	0.0012221129 0.0000006660 0.0012214469	0.0003589683 0.0000066988 0.0003522695

Appendix

The approximate mean and variance of inverted Gompertz distribution

If $Y \sim \mathrm{EXP}(\mathrm{a})$, with $\mu = E(y) = \frac{1}{\mathrm{a}}$ and $\sigma^2 = V(y) = \frac{1}{\mathrm{a}^2}$, then the variable $T = g(Y) = \left[\frac{1}{b}\ln(1+by)\right]^{-1} \sim \mathrm{IG}(\mathrm{a}, b)$. This relation will be used to find the approximate mean and variance of $\mathrm{IG}(\mathrm{a}, b)$. The approximate mean and variance of the $g(y)$, based on the method of statistical differentials, are given by

$$E[g(Y)] \cong g(\mu) + \frac{1}{2}\sigma^2 g''(\mu), \tag{A.1}$$

and

$$\mathrm{Var}[g(Y)] \cong \sigma^2 (g'(\mu))^2. \tag{A.2}$$

Now $$T = g(Y) = \left[\frac{1}{b}\ln(1+by)\right]^{-1},$$

then

$$g(\mu) = \left[\frac{1}{b}\ln(1+b\mu)\right]^{-1}, \quad g'(\mu) = -(1+b\mu)^{-1}\left[\frac{1}{b}\ln(1+b\mu)\right]^{-2}, \tag{A.3}$$

and

$$g''(\mu) = (1+b\mu)^{-2}\left[\frac{1}{b}\ln(1+b\mu)\right]^{-3}\left[2+\ln(1+b\mu)\right], \tag{A.4}$$

where $\mu = \frac{1}{\mathrm{a}}$ and $\sigma^2 = \frac{1}{\mathrm{a}^2}$.

By substituting (A.3) and (A.4) in (A.1) and (A.2), we obtain

$$E[g(y)] \cong \left[\frac{1}{b}\ln\left(1+\frac{b}{\mathrm{a}}\right)\right]^{-3}\left\{\left[\frac{1}{b}\ln\left(1+\frac{b}{\mathrm{a}}\right)\right]^{2} + \frac{1}{2\mathrm{a}^2}\left(1+\frac{b}{\mathrm{a}}\right)^{-2}\left[2+\ln\left(1+\frac{b}{\mathrm{a}}\right)\right]\right\}, \tag{A.5}$$

and

$$V[g(y)] \cong \frac{1}{\mathrm{a}^2}\left(1+\frac{b}{\mathrm{a}}\right)^{-2}\left[\frac{1}{b}\ln\left(1+\frac{b}{\mathrm{a}}\right)\right]^{-4}. \tag{A.6}$$

1

REFERENCES

Adham, S. A. (1996). *On finite mixture of two-component Gompertz lifetime model*, Master's thesis, King Abdulaziz University, Jeddah-Kingdom of Saudia Arabia.

Adham, S. A. and Walker, S. G. (2001). A multivariate Gompertz-Type distribution. *J. Appl. Statist.*, Vol. 28, Issue 8, pp. 1051-1065.

Ahuja, J. C. and Nash, S. W. (1967). The generalized Gompertz-Verhulst family of distributions. *Sankhya*, Part A, Vol. 29, pp. 141-156.

AL-Hussaini, E. K., AL-Dayian, G. R. and Adham, S. A. (2000). On finite mixture of two-component Gompertz lifetime model. *J. Statist. Comput. Simul.*, Vol. 67, pp. 1-20.

AL-Khedhairi, A. and EL-Gohary, A. (2008). A new class of bivariate Gompertz distributions and its mixture. *Int. Journal of Math. Analysis*, Vol. 2, No. 5, pp. 235-253.

Garg, M. L., Raja Rao, B. and Redmond, C. K. (1970). Maximum-likelihood estimation of parameters of the Gompertz survival function. *J. Roy. Statist. Soc., Ser. Appli. Statisti.*, Vol. 19, pp. 152-159.

Hendi, M. I., Abu-Youssef, S. E. and Alraddadi, A. A. (2006). Order statistics from truncated Gompertz distribution and its characterizations. *The Egyptian Statistical Journal, ISSR, Cairo University*, Vol. 50, No. 1, pp. 21-31.

Jaheen, Z. F. (2003a). A Bayesian analysis of record statistics from the Gompertz model. *Appli. Mathe. Comput.*, Vol. 145, pp. 307-320.

Jaheen, Z. F. (2003b). Bayesian prediction under a mixture of two-component Gompertz lifetime model. *Test*, Vol. 12, No. 2, pp. 413-426.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). *Continuous univariate distributions*, Vol. 1, 2nd ed., John Wiley and Sons, Inc., New York.

Sherman, J. and Morrison, W. J. (1950). Simplified procedures for fitting a Gompertz curve and a modified exponential curve. *J. Amer. Statist. Assoc.*, Vol. 45, pp. 87-98.

Wu, J. W., Hung, W. L. and Tsai, C. H. (2004). Estimation of parameters of the Gompertz distribution using the least squares method. *Appli. Mathe. Comput.*, Vol. 158, pp. 133-147.