Maximum Likelihood Estimation of the Flexible Weibull Distribution Based on Type II censored Data*

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Abstract

Based on type II censored sample, the maximum likelihood estimators of the parameters of an extension of the Weibull distribution referred to as the flexible Weibull distribution are obtained. Numerical results, based on simulated data, as well as concluding remarks regarding the results are provided.

Key Words: *Flexible Weibull; Maximum likelihood estimator; Type-II censoring; Variance-covariance matrix.*

1. Introduction

The basic two parameter Weibull distribution given by the distribution function:

$$F(t) = 1 - e^{-\left(\frac{t}{\beta}\right)^{\alpha}}, \qquad t > 0; \quad \alpha, \beta > 0 \qquad (1.1)$$

is one of, if not, the most popular statistical distributions. It is utmost interest to both theoretical statistics and practitioners. Its density function changes rapidly as the shape parameter α changes from values less than one to values greater than one. It is a generalization of the exponential distribution and a special case of the extreme value distribution [See Johnson *et al* (1994)]. It is of interest to practitioners because of its ability to fit data from various fields of application. An interesting characteristic of the Weibull distribution is its failure rate [See Lai and Xie(2006)]. Weibull distributions with $\alpha < 1$ have a failure rate that decreases with time. Weibull distributions with α close or equal to one have a fairly constant failure rate. Weibull distributions with $\alpha > 1$ have a failure rate that increases with time. However, over many years and across a wide variety of mechanical and electronic systems people have calculated empirical population failure rate as units of age and repeatedly obtained graphs of failure rates that exhibit all three cases mentioned above. Because of the shape of a curve with the three characteristics in sequence with the decreasing failure rate in early life followed by a fairly constant failure rate in midlife, and then an increasing failure rate, it has become known as the "bathtub" failure rate [Hong and Diew (2007)]. In searching for statistical distribution to model bathtub failure rate research for most part have tried to stay close to the Weibull distributions. Many extension generalization and modifications of the Weibull distributions have been introduced to give bathtub failure rate [Xie *et al*. (2002)].
If the limit of the failure rate at time zero is zero, the failure rate has been described as modified bathtub shape (MBT). So, the difference between the traditional and the modified bathtub curves is the behaviour during the infant mortality phase [Lai and Xie (2006)].

Gurvich *et al*. (1997) introduced a class of distributions characterized by the cumulative distribution function:

, t > 0 , (1.2)

with parameter a > 0, where G (t) is a monotonically increasing function of t, G (t) >0. Gurvich *et al.* (1997) considered a class of distributions generalized the traditional Weibull model. They considered (1.2) as a generalized Weibull distribution and when G (t) takes different formulas different distributions can be obtained.

Lai *et al*. (2003) reviewed several families of extended Weibull distributions, while a comprehensive taxonomy of Weibull models can be found in Murthy *et al*. (2004) .They discussed additional applications, and gave a methodological review of the 'Weibull area'. They also suggested further study of various Weibull-type distributions and model selection. Nadarajah and Kotz (2005) showed that several existing life distributions may be expressed in the form (1.2). Another modification of the Weibull distribution was introduced by Bebbington *et al.* (2006); this distribution is an extension of the Weibull distribution. The distribution is another member of the Weibull family, which they named the 'flexible Weibull' distribution. It has not a monotonic failure rate function of t, but it is the class of lifetime distributions which has a bathtub shaped failure rate function. It is very important because the lifetime of electronic, electromechanical, and mechanical products are often modeled with this class. This distribution includes the bathtub shape, the modified bathtub shape and other different shapes for failure rate. So this distribution is quite flexible.

Let T be a lifetime random variable with a density function given by:

), t > 0; α, β>0 , (1.3)

and a cumulative distribution function given by:

F(t) = 1−), t > 0. (1.4)

The reliability function and the failure rate function of this two-parameter flexible Weibull distribution are given respectively by:

), $\quad t > 0; \quad \alpha, \beta > 0,$ (1.5)

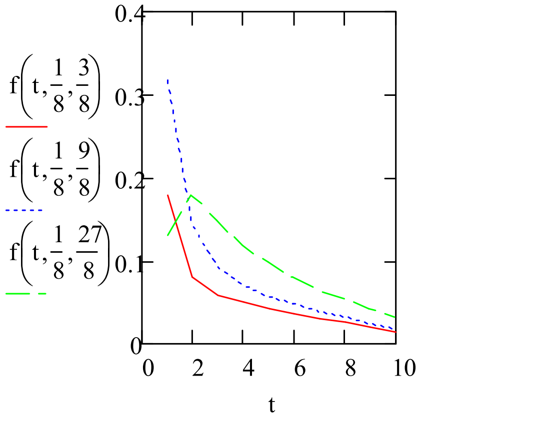
$h(t) =$, $\quad t > 0; \quad \alpha, \beta > 0.$ (1.6)

The formulas for the mean and the variance are difficult to obtain in closed form, the quantiles are easy to evaluate, Let t_p be the pth quantile of T. By considering the log–log transformation of p = R (t), we have log (-log p) $= (\alpha\, t^2 - \beta)/t$, and so t is a solution of the quadratic equation $\alpha t^2 - \log(-\log p)\, t - \beta = 0$, Since the solutions have to be non-negative, the only one is:

$$t_p = \frac{1}{2\alpha \left(\log\left(-\log[p] \right) + \sqrt{\{\log(-\log[p])\}^2 + 4\alpha\beta} \right)}$$) .

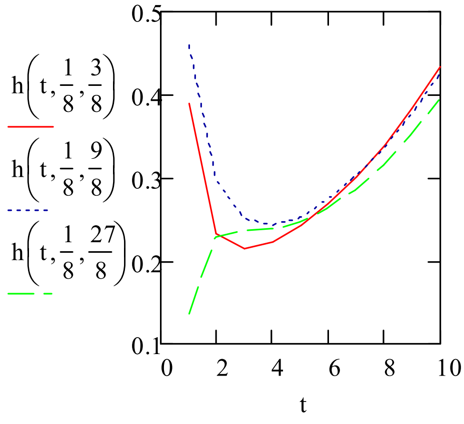
The curves of the pdf and h (t) of the flexible Weibull distributions with parameters α, β are plotted in Figure (1).

Figure (1)



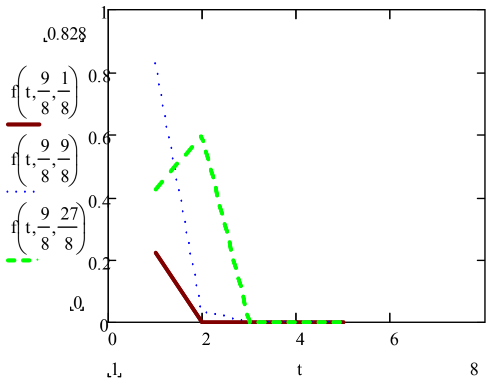
The probability density function when α = 1/8, β = 3/8, 9/8, 27/8

(a)



The failure rate function when α = 1/8, β = 3/8, 9/8, 27/8

(b)



The probability density function when α = 9/8, β = 1/8, 9/8, 27/8

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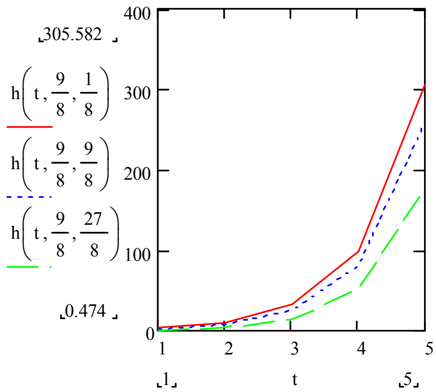

The failure rate function when α = 9/8, β = 1/8, 9/8, 27/8

(d)

The shapes of the density and failure rate functions are illustrated for selected values of α and β in Figure (1). We see that as β decreases, the failure rate function becomes more 'bathtub-like', while, as α increase, the 'bathtub' becomes 'shallower'.

Bebbington *et al.* (2006) showed that the failure rate function h (t) is increasing if and only if $\alpha\,\beta > 27/64$, when $\alpha\,\beta < 27/64$, the distribution is MBT.

The maximum likelihood estimate for the parameters for type II censored sample data are obtained in Section 2. In Section 3 Monte Carlo simulations are studied, also concluding remarks are given in Section 3.

2. Maximum likelihood Estimators based on type II censored

Suppose that $t_1<t_2<........<t_k$ is a type II censored sample of size K obtained from a life test of n items whose lifetimes has a flexible Weibull distribution with two parameters, α and β.

The likelihood function based on the type II censored sample is

$$L(\alpha,\beta,t) = \frac{n!}{(n-k)}\prod_{i=1}^{n} f(t_i)(1-F[(t_k)])^{n-k} \quad , \qquad (2.1$$

(t) are given by Equations (1.3) and (1.4) respectively .where f (t) and F

Substituting (1.3) and (1.4) in (2.1) one obtains:

L(α, β, t) = n!/((n − k)) Π↓(i = 1)↑k █((α + (β)/ t↓i ↑2) exp(α∑↓(i = 1)↑k t↓i − β∑↓(i = 1)↑k 1/t↓i) exp

The natural logarithm of the likelihood function, in Equation (2.2) is given by:

$l = \ln L(t, \alpha, \beta)$

$$l = \ln n! - \ln(n-k)! + \sum_{i=1}^{k} \ln\left(\alpha + \frac{\beta}{t_i^{\,2}}\right) + \alpha \sum_{i=1}^{k} t_i - \sum_{i=1}^{k} \frac{\beta}{t_i}$$

Differentiating (2.3) with respect to α and β and setting to zero, we obtain:

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{k} \frac{t_i^2}{\alpha t_i^2 + \beta} + \sum_{i=1}^{k} t_i - \sum_{i=1}^{k} t_i e^{\alpha t_i - \frac{\beta}{t_i}} - (n-k) t_k e^{\alpha t_k - \frac{\beta}{t_k}} = 0, \qquad (2.4)$$

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{k} \frac{1}{\alpha t_i^2 + \beta} - \sum_{i=1}^{k} \frac{1}{t_i} + \sum_{i=1}^{k} \frac{1}{t_i} e^{\alpha t_i - \frac{\beta}{t_i}} + (n-k) \frac{1}{t_k} e^{\alpha t_k - \frac{\beta}{t_k}} = 0, \qquad (2.5)$$

The maximum likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$ can then be obtained by solving the two non linear likelihood Equations (2.4), (2.5).

The variance-covariance matrix of MLE's is given by the elements of the inverse of the fisher information matrix.

$$I(\alpha, \beta) = -E \begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} \\ \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \beta^2} \end{pmatrix}, \qquad (2.6)$$

Unfortunately, the exact mathematical expressions for the above expectations are very difficult to obtain, therefore, the observed fisher information matrix is used which is obtained by dropping the (α,β)[Cohen(1965)].$\hat{I}$expectation operator E so we obtain the approximate fisher information matrix

The sample information matrix is used in the numerical study, which is defined as follows

$$\hat{I}(\alpha,\beta) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} \\ \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \beta^2} \end{pmatrix}_{\alpha=\hat{\alpha},\ \beta=\hat{\beta}} , \qquad (2.7)$$

covariance matrix is obtained by inverting the observed information matrix as follows –The Variance

$$\hat{V} = \left[\hat{I}(\alpha,\beta)\right]^{-1} , \qquad (2.8)$$

where

$$\hat{V}(\hat{\alpha}) = -\frac{\partial^2 l}{\partial \alpha^2} , \qquad (2.9)$$

$$\hat{V}(\hat{\beta}) = -\frac{\partial^2 l}{\partial \beta^2} , \qquad (2.10)$$

and

$$\hat{\text{cov}}(\hat{\alpha}) = -\frac{\partial^2 l}{\partial \beta \partial \alpha} , \qquad (2.11)$$

$$\hat{\text{cov}}(\hat{\beta}) = -\frac{\partial^2 l}{\partial \alpha \partial \beta} . \qquad (2.12)$$

The elements of the information matrix

Differentiating (2.4) and (2.5) with respect to α and β gives:

$$\frac{\partial^2 l}{\partial \alpha^2} = \sum_{i=1}^{k} \frac{-(t_i^2)^2}{(\alpha t_i^2 + \beta)^2} - \sum_{i=1}^{n} t_i^2 e^{\alpha t_i - \frac{\beta}{t_i}} - (n-k) t_k^2 e^{\alpha t_k - \frac{\beta}{t_k}}, \qquad (2.13)$$

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = \sum_{i=1}^{k} \frac{-t_i^2}{(\alpha t_i^2 + \beta)^2} + \sum_{i=1}^{k} e^{\alpha t_i - \frac{\beta}{t_i}} + (n-k) e^{\alpha t_k - \frac{\beta}{t_k}} , \qquad (2.14)$$

$$\frac{\partial^2 l}{\partial \beta^2} = \sum_{i=1}^{k} \frac{-1}{\left(\alpha t_i^2 + \beta\right)^2} - \sum_{i=1}^{k} \left(\frac{1}{t_i}\right)^2 e^{\alpha t_i - \frac{\beta}{t_i}} - (n-k)\left(\frac{1}{t_k}\right)^2 e^{\alpha t_k - \frac{\beta}{t_k}} \quad . \qquad (2.15)$$

The MLE's of the parameters of the flexible Weibull distribution based on complete data is the special case from type II censored sample data when k = n

3. Simulation Study and Concluding Remarks

The population parameters are generated using the mathematical computing package Math Cad (14) to yield the MLE's of the parameters α, β in two cases: complete and type II censored sample data.

The simulation results are obtained according to the following steps:

1- Obtain an initial value of α, β.

2- Given α and β, generate a random sample of size [n=10, 20, 30, 50, 100] from N=10000 replication for different values of the parameters α, β. and different censoring levels [P=0.20, 0.30, 0.50].

3- From the flexible Weibull distribution which defined in equation (1-3), this can be achieved by generating a random sample from the uniform (0, 1) distribution. Then, the uniform random variables can be transformed to the flexible Weibull random variables using the following transformation,

$$1 - e^{-e^{\alpha t - \frac{\beta}{t}}} = \square_{i,j} .$$

4 - Solve the non linear equation (2.4) and (2.5) together to obtain $\hat{\alpha}$ and $\hat{\beta}$.

The maximum likelihood estimates for the parameters, the bias, the relative absolute bias, the mean square error, the relative mean square error and the variance have been calculated.

The results obtained for complete sample data are displayed in Tables (1-3) and the results obtained for type II censored sample data are displayed in Tables (4-11).

From Tables (1-3) it is observed that:

1. The maximum likelihood estimates for the parameters perform better when the sample size n increased.
2. It is also observed that the bias, the relative absolute bias, the mean square error, the relative mean square error and the variance became better for large samples.

From Tables (4-11) it is observed that:

1. The estimates for the parameters perform better when the sample size n increased.

2. Better estimates are obtained when the level of censoring decreased and this remark is expected since decreasing the level of censoring means that more information is provided by the sample and hence increases the accuracy of the estimates.

In general in the case of complete data and censored data we note that:

When $\alpha > \beta$ and the value of β is near to zero the values of all estimates are decreasing for large samples and we note that the values of $\hat{\alpha}$ is fare from the true value of the parameter while the values of $\hat{\beta}$ is near to the true value of the parameter. [See Table (1) in the case of complete data and Table (10, 11) in the case of type II censored].

The effective sample size n and the number of failures k play more important role regarding the efficiency of the maximum likelihood estimator.

Table (1): bias, Relative Absolute bias (Rel.A.B), Mean Square Error(MSE), Relative Mean Square Error (Rel.MSE), variance (VAR) and Sample Information (S.Infor) of the MLE's for complete Sample at N=10000, α=1/2 β=1/8,3/8,9/8,27/8,

n	parameter	estimator	$\hat{\beta}, \hat{\alpha}$ Bias_	Rel.A.B	MSE	Rel.MSE	VAR.	S . Infor.
10	α =0.5	=0.429 $\hat{\alpha}$	-0.071	0.143	0.026	0.053	0.021	$\begin{pmatrix} 9.518\times10^{-3} & -1.384\times10^{-4} \\ -1.384\times10^{-4} & 0.011 \end{pmatrix}$
	β =0.125	=0.418 $\hat{\beta}$	0.293	2.344	0.153	1.222	0.067	
	α =0.5	=0.648 $\hat{\alpha}$	0.148	0.296	0.123	0.245	0.101	$\begin{pmatrix} 0.018 & 3.027\times10^{-3} \\ 3.027\times10^{-3} & 0.024 \end{pmatrix}$
	β =.375	=0446 $\hat{\beta}$	0.071	0.19	0.045	0.119	0.04	
	α=0.5	=0.617 $\hat{\alpha}$	0.117	0.234	0.062	0.125	0.049	$\begin{pmatrix} 0.016 & 0.022 \\ 0.022 & 0.183 \end{pmatrix}$
	β=1.125	=1.338 $\hat{\beta}$	0.213	0.19	0.311	0.277	0.266	
	α=0.5	=0.595 $\hat{\alpha}$	0.095	0.19	0.044	0.089	0.035	$\begin{pmatrix} 3.923\times10^{-8} & 9.033\times10^{-5} \\ 9.033\times10^{-5} & 0.208 \end{pmatrix}$
	β=3.375	=3. 963 $\hat{\beta}$	0.588	0.174	2.152	0.638	1.806	
n=20	α=0.5	=0.402 $\hat{\alpha}$	-0.098	0196	0.016	0.032	$\times 10^{-3}$ 6.29	$\begin{pmatrix} 9.518\times10^{-3} & -1.384\times10^{-4} \\ -1.384\times10^{-4} & 0.011 \end{pmatrix}$
	β=0.125	=0.366 $\hat{\beta}$	0.241	1.93	0.073	0.583	0.015	
	α =0.5	=0.561 $\hat{\alpha}$	0.061	0.122	0.026	0.052	0.022	$\begin{pmatrix} 0.01 & 9.667\times10^{-4} \\ 9.667\times10^{-4} & 0.01 \end{pmatrix}$
	β =0.375	=0.409 $\hat{\beta}$	0.034	0.089	0.1014	0.038	0.013	
	α=0.5	=0.55 $\hat{\alpha}$	0.05	0.101	0.017	0.033	0.014	$\begin{pmatrix} 8.411\times10^{-3} & 9.154\times10^{-3} \\ 9.154\times10^{-3} & 0.079 \end{pmatrix}$
	β=1.125	=1.221 $\hat{\beta}$	0.096	0.085	0.103	0.092	0.094	
	α=0.5	=0.535 $\hat{\alpha}$	0.035	0.069	0.016	0.032	0.015	$\begin{pmatrix} 5.444\times10^{-8} & 1.254\times10^{-4} \\ 1.254\times10^{-4} & 0.289 \end{pmatrix}$
	β=3.375	=3.63 $\hat{\beta}$	0.255	0.076	0.753	0.223	0.688	

Table (2): bias, Relative Absolute bias (Rel.A.B), Mean Square Error(MSE), Relative Mean Square Error (Rel.MSE), and Sample Information (S.Infor) of the MLE's for complete Sample at N=10000, α=1/٢, β=1/8,3/8,9/8,27/8

n	parameter	estimator	$\hat{\beta},\hat{\alpha}$ Bias_	Rel.A.B	MSE	Rel.MSE	VAR.	**S** Infor**.**
30	α =0.5	=0.394 $\hat{\alpha}$	-0.106	0.212	0.015	0.03	3.677 $\times 10^{-3}$	
	β =0.125	=0.355 $\hat{\beta}$	0.23	1.838	0.06	0.477	6.824 $\times 10^{-3}$	$\begin{pmatrix} 6.189 \times 10^{-3} & -1.281 \times 10^{-4} \\ -1.281 \times 10^{-4} & 6.75 \times 10^{-3} \end{pmatrix}$
	α =0.5	=0.538 $\hat{\alpha}$	0.038	0.076	0.012	0.025	0.011	
	β =0.375	=0.396 $\hat{\beta}$	0.021	0.057	$\times 10^{-3}$ 8.07	0.022	7.612 $\times 10^{-3}$	$\begin{pmatrix} 7.113 \times 10^{-3} & 5.276 \times 10^{-4} \\ 5.276 \times 10^{-4} & 6.612 \times 10^{-3} \end{pmatrix}$
	α=0.5	=0.532 $\hat{\alpha}$	0.032	0.063	$\times 10^{-3}$ 8.905	0.018	7.903 $\times 10^{-3}$	
	β=1.125	=1.183 $\hat{\beta}$	0.058	0.052	0.059	0.053	0.056	$\begin{pmatrix} 5.685 \times 10^{-3} & 5.67 \times 10^{-3} \\ 5.67 \times 10^{-3} & 0.05 \end{pmatrix}$
	α=0.5	=0.517 $\hat{\alpha}$	0.017	0.034	0.013	0.026	0.013	
	β=3.375	=3.53 $\hat{\beta}$	0.155	0.046	0.447	0.132	0.423	$\begin{pmatrix} 3.816 \times 10^{-8} & 5.67 \times 10^{-3} \\ 5.67 \times 10^{-3} & 0.202 \end{pmatrix}$
50	α=0.5	=0.39 $\hat{\alpha}$	-0.11	0.219	0.014	0.028	2.035 $\times 10^{-3}$	
	β=0.125	=0.344 $\hat{\beta}$	0.219	1.752	0.051	0.408	$\times 10^{-3}$ 3.06	$\begin{pmatrix} 63.699 \times 10^{-3} & -9.192 \times 10^{-5} \\ -9.192 \times 10^{-5} & 3.819 \times 10^{-3} \end{pmatrix}$
	α =0.5	=0.522 $\hat{\alpha}$	0.022	0.043	$\times 10^{-3}$ 6.075	0.012	5.605 $\times 10^{-3}$	
	β =0.375	=0.387 $\hat{\beta}$	0.012	0.033	$\times 10^{-3}$ 4.341	0.012	$\times 10^{-3}$ 4.19	$\begin{pmatrix} 4.329 \times 10^{-3} & 2.641 \times 10^{-4} \\ 2.641 \times 10^{-4} & 3.807 \times 10^{-3} \end{pmatrix}$
	α=0.5	=0.518 $\hat{\alpha}$	0.018	0.036	$\times 10^{-3}$ 4.57	$\times 10^{-3}$ 9.139	4.244 $\times 10^{-3}$	

	β=1.125	$=1.159\hat{\beta}$	0.034	0.03	0.032	0.028	0.031	
	α=0.5	$=0.497\hat{\alpha}$	-3.122×10^{-3}	6.244×10^{-3}	0.013	0.027	0.013	
	β=3.375	$=3.439\hat{\beta}$	0.064	0.019	0.261	0.077	0.257	$\begin{pmatrix} 2.35 \times 10^{-8} & 5.41 \times 10^{-5} \\ 3.22 \times 10^{-5} & 0.125 \end{pmatrix}$
100	α =0.5	$=0.385\hat{\alpha}$	-0.115	0.229	0.014	0.028	9.707×10^{-4}	
	β =0.125	$=0.355\hat{\beta}$	0.23	1.838	0.06	0.477	1.385×10^{-3}	$\begin{pmatrix} 1.816 \times 10^{-3} & -5.112 \times 10^{-5} \\ -5.112 \times 10^{-5} & 1.87 \times 10^{-3} \end{pmatrix}$
	α =0.5	$=0.509\hat{\alpha}$	9.354×10^{-3}	0.019	2.586×10^{-3}	5.172×10^{-3}	2.499×10^{-3}	
	β =0.375	$=0.38\hat{\beta}$	5.426×10^{-3}	0.014	1.988×10^{-3}	5.302×10^{-3}	1.959×10^{-3}	$\begin{pmatrix} 2.181 \times 10^{-3} & 1.131 \times 10^{-4} \\ 1.131 \times 10^{-4} & 1.845 \times 10^{-3} \end{pmatrix}$
	α=0.5	$=0.508\hat{\alpha}$	8.363×10^{-3}	0.017	2.028×10^{-3}	4.056×10^{-3}	1.958×10^{-3}	
	β=1.125	$=1.141\hat{\beta}$	0.016	0.014	0.015	0.013	0.015	$\begin{pmatrix} 1.724 \times 10^{-3} & 1.543 \times 10^{-3} \\ 1.543 \times 10^{-3} & 0.014 \end{pmatrix}$
	α=0.5	$=0.468\hat{\alpha}$	-0.032	0.065	0.021	0.041	0.019	
	β=3.375	$=3.344\hat{\beta}$	-0.031	9.056×10^{-3}	0.177	0.052	0.176	$\begin{pmatrix} 1.202 \times 10^{-8} & 2.769 \times 10^{-5} \\ 2.769 \times 10^{-5} & 0.064 \end{pmatrix}$

Table (3): bias, Relative Absolute bias (Rel.A.B), Mean Square Error(MSE), Relative Mean Square Error (Rel.MSE), and Sample Information (S.Infor) of the MLE's for complete Sample at N=10000, at α=9/8, β=3/8,9/8,27/8, n=10,20,30,50,100

	parameter	Estimator	$\hat{\beta},\hat{\alpha}$ Bias_	R_abs	MSE	R_mse	VAR.	S Infor.
10	α =1.125	=1.404$\hat{\alpha}$	0.279	0.248	0.364	0.323	0.286	
	β =0.375	=0.447$\hat{\beta}$	0.072	0.192	0.037	0.099	0.032	$\begin{pmatrix} 0.082 & 0.014 \\ 0.014 & 0.022 \end{pmatrix}$
	α =1.125	=1.356$\hat{\alpha}$	0.231	0.206	0.233	0.207	0.179	
	β =1.125	=1.328$\hat{\beta}$	0.203	0.18	0.256	0.228	0.215	$\begin{pmatrix} 0.076 & 0.014 \\ 0.069 & 0.152 \end{pmatrix}$
	α =1.125	=1.326$\hat{\alpha}$	0.201	0.178	0.213	0.189	0.173	
	β =3.375	=3.943$\hat{\beta}$	0.568	0.168	2.121	0.628	1.799	$\begin{pmatrix} 9.117\times10^{-15} & 1.039\times10^{-11} \\ 1.039\times10^{-11} & 1.184\times10^{-8} \end{pmatrix}$
20	α =1.125	=1.244$\hat{\alpha}$	0.119	0.106	0.094	0.083	0.08	
	β =0.375	=0.407$\hat{\beta}$	0.032	0.086	0.012	0.033	0.011	$\begin{pmatrix} 0.045 & 5.569\times10^{-3} \\ 5.569\times10^{-3} & 9.279\times10^{-3} \end{pmatrix}$
	α =1.125	=1.226$\hat{\alpha}$	0.101	0.09	0.066	0.059	0.056	
	β =1.125	=1.217$\hat{\beta}$	0.092	0.081	0.087	0.077	0.079	$\begin{pmatrix} 0.038 & 0.03 \\ 0.013 & 0.067 \end{pmatrix}$
	α =1.125	=1.202$\hat{\alpha}$	0.077	0.068	0.08	0.071	0.074	$\begin{pmatrix} 3.849\times10^{-13} & 4.387\times10^{-10} \\ 4.387\times10^{-10} & 4.999\times10^{-7} \end{pmatrix}$

	0.746	0.236	0.797	0.067	0.225	$\hat{\beta}=3.6$	$\beta=3.375$	
	0.044	0.044	0.049	0.064	0.072	$\hat{\alpha}=1.197$	$\alpha=1.125$	30
$\begin{pmatrix} 0.03 & 3.395\times10^{-3} \\ 3.395\times10^{-3} & 5.9\times10^{-3} \end{pmatrix}$	6.622×10^{-3}	0.019	7.042×10^{-3}	0.055	0.02	$\hat{\beta}=0.395$	$\beta=0.375$	
	0.033	0.033	0.037	0.057	0.064	$\hat{\alpha}=1.189$	$\alpha=1.125$	
$\begin{pmatrix} 0.025 & 0.019 \\ 0.0119 & 0.043 \end{pmatrix}$	0.047	0.045	0.05	0.05	0.056	$\hat{\beta}=1.181$	$\beta=1.125$	
	0.068	0.061	0.069	0.025	0.028	$\hat{\alpha}=1.153$	$\alpha=1.125$	
$\begin{pmatrix} 1.172\times10^{-12} & 1.336\times10^{-9} \\ 1.336\times10^{-9} & 1.523\times10^{-6} \end{pmatrix}$	0.582	0.176	0.592	0.03	0.1	$\hat{\beta}=3.475$	$\beta=3.375$	
	0.022	0.022	0.024	0.039	0.044	$\hat{\alpha}=1.169$	$\alpha=1.125$	50
$\begin{pmatrix} 0.019 & 1.931\times10^{-3} \\ 1.931\times10^{-3} & 3.436\times10^{-3} \end{pmatrix}$	3.653×10^{-3}	0.01	3.831×10^{-3}	0.036	0.013	$\hat{\beta}=0.388$	$\beta=0.375$	
	0.018	0.017	0.019	0.032	0.036	$\hat{\alpha}=1.161$	$\alpha=1.125$	
$\begin{pmatrix} 0.015 & 0.011 \\ 0.011 & 0.025 \end{pmatrix}$	0.026	0.024	0.027	0.029	0.033	$\hat{\beta}=1.158$	$\beta=1.125$	
	0.011	9.83×10^{-3}	0.011	0.017	0.02	$\hat{\alpha}=1.145$	$\alpha=1.125$	100
$\begin{pmatrix} 9.315\times10^{-3} & 9.04\times10^{-4} \\ 9.04\times10^{-4} & 1.661\times10^{-3} \end{pmatrix}$	1.732×10^{-3}	4.698×10^{-3}	1.762×10^{-3}	0.014	5.429×10^{-3}	$\hat{\beta}=0.38$	$\beta=0.375$	
	8.367×10^{-3}	7.691×10^{-3}	8.653×10^{-3}	0.015	0.017	$\hat{\alpha}=1.142$	$\alpha=1.125$	
[illegible]	0.013	0.011	0.013	0.014	0.016	$\hat{\beta}=1.141$	$\beta=1.125$	

Table (4): (A.B), Mean Square Error (MSE), variance (VAR) and Sample Information (S.Infor) of the MLE's Based on type II Censored Sample, at $\alpha=1/8$, $\beta=3/8$ n=10,20,30

S Infor.	VAR	MSE	Bias_ $\hat{\alpha}$, $\hat{\beta}$	Estimator	p	n
$\begin{pmatrix} 1.775\times10^{-3} & 6.288\times10^{-4} \\ 6.288\times10^{-4} & 0.028 \end{pmatrix}$	0.259 0.063	0.307 0.068	0.22 0.075	$\hat{\alpha}$ =0.345 $\hat{\beta}$ =0.45	0.8	10
$\begin{pmatrix} 1.943\times10^{-3} & 9.933\times10^{-4} \\ 9.933\times10^{-4} & 0.028 \end{pmatrix}$	0.644 0.072	0.786 0.077	0.378 0.068	$\hat{\alpha}$ =0.503 $\hat{\beta}$ =0.443	0.7	
$\begin{pmatrix} 2.469\times10^{-4} & 1.028\times10^{-3} \\ 1.028\times10^{-3} & 0.028 \end{pmatrix}$	0.6446 0.089	0.8198 0.113	1.324 0.154	$\hat{\alpha}$ =1.449 $\hat{\beta}$ =0.529	0.5	
$\begin{pmatrix} 2.495\times10^{-3} & -8.048\times10^{-6} \\ -8.048\times10^{-6} & 0.011 \end{pmatrix}$	0.034 0.015	0.04 0.016	0.076 0.03	$\hat{\alpha}$ =0.201 $\hat{\beta}$ =0.405	0.8	20
$\begin{pmatrix} 4.865\times10^{-3} & -3.111\times10^{-4} \\ 3.111\times10^{-4} & 0.011 \end{pmatrix}$	0.094 0.023	0.118 0.023	0.154 0.017	$\hat{\alpha}$ =0.279 $\hat{\beta}$ =0.392	0.7	
$\begin{pmatrix} 0.03 & 4.232\times10^{-3} \\ 4.232\times10^{-3} & 0.013 \end{pmatrix}$	0.657 0.018	0.906 0.021	0.499 0.059	$\hat{\alpha}$ =0.624 $\hat{\beta}$ =0.434	0.5	

$\begin{pmatrix} 1.944\times10^{-3} & -1.349\times10^{-4} \\ -1.349\times10^{-4} & 6.982\times10^{-3} \end{pmatrix}$	9.771×10^{-3} 8.341×10^{-3}	0.012 8.653×10^{-3}	0.043 0.018	$\hat{\alpha}$ =0.168 $\hat{\beta}$ =0.393	0.8	30
$\begin{pmatrix} 4.378\times10^{-3} & -2.067\times10^{-5} \\ -2.067\times10^{-5} & 6.861\times10^{-3} \end{pmatrix}$	0.74 0.022	0.043 0.013	0.093 8.232×10^{-3}	$\hat{\alpha}$ =0.218 $\hat{\beta}$ =0.383	0.7	
$\begin{pmatrix} 0.039 & 3.173\times10^{-3} \\ 3.173\times10^{-3} & 8.027\times10^{-3} \end{pmatrix}$	0.258 9.396×10^{-3}	0.349 0.029	0.301 0.036	$\hat{\alpha}$ =0.426 $\hat{\beta}$ =0.411	0.5	

Table (5): (A.B), Mean Square Error (MSE), variance (VAR) and Sample Information (S.Infor) of the MLE's Based on type II Censored Sample, at α=1/8, β=3/8 n=50,100

S Infor.	VAR	MSE	Bias_ $\hat{\alpha}$, $\hat{\beta}$	Estimator	p	n
$\begin{pmatrix} 1.249\times10^{-3} & -1.432\times10^{-4} \\ -1.432\times10^{-4} & 4.003\times10^{-3} \end{pmatrix}$	3.337×10^{-3} 4.296×10^{-3}	3.849×10^{-3} 4.392×10^{-3}	0.023 9.818×10^{-3}	$\hat{\alpha}$ =0.148 $\hat{\beta}$ =0.385	0.8	50
$\begin{pmatrix} 3.107\times10^{-3} & -1.663\times10^{-4} \\ -1.663\times10^{-4} & 4.022\times10^{-3} \end{pmatrix}$	0.012 6.755×10^{-3}	0.015 6.806×10^{-3}	0.05 7.147×10^{-3}	$\hat{\alpha}$ =0.175 $\hat{\beta}$ =0.382	0.7	
$\begin{pmatrix} 0.035 & 1.713\times10^{-3} \\ 1.713\times10^{-3} & 4.427\times10^{-3} \end{pmatrix}$	0.114 4.92×10^{-3}	0.147 5.361×10^{-3}	0.183 0.021	$\hat{\alpha}$ =0.308 $\hat{\beta}$ =0.396	0.5	
$\begin{pmatrix} 6.383\times10^{-4} & -9.176\times10^{-5} \\ -9.176\times10^{-5} & 1.951\times10^{-3} \end{pmatrix}$	1.045×10^{-3} 2.034×10^{-3}	1.148×10^{-3} 2.065×10^{-3}	0.01 5.551×10^{-3}	$\hat{\alpha}$ =0.135 $\hat{\beta}$ =0.381	0.8	100
$\begin{pmatrix} 1.67\times10^{-3} & -1.488\times10^{-4} \\ -1.488\times10^{-4} & 1.949\times10^{-3} \end{pmatrix}$	3.528×10^{-3} 2.605×10^{-3}	4.063×10^{-3} 2.617×10^{-3}	0.023 3.389×10^{-3}	$\hat{\alpha}$ =0.148 $\hat{\beta}$ =0.378	0.7	
$\begin{pmatrix} 0.021 & 5.874\times10^{-4} \\ 5.874\times10^{-4} & 2.053\times10^{-3} \end{pmatrix}$	0.047 2.374×10^{-3}	0.054 2.455×10^{-3}	0.085 8.983×10^{-3}	$\hat{\alpha}$ =0.21 $\hat{\beta}$ =0.384	0.5	

Table (6): (A.B), Mean Square Error (MSE), variance (VAR) and Sample Information (S.Infor) of the MLE's Based on type II Censored Sample, at α=1/2, β=9/8 n=10,20,30,

S. Infor..	VAR	MSE	Bias_ $\hat{\alpha}$, $\hat{\beta}$	Estimator	p	n
$\begin{pmatrix} 0.035 & 0.045 \\ 0.045 & 0.21 \end{pmatrix}$	0.162 0.395	0.209 0.467	0.218 0.268	$\hat{\alpha}$ =0.718 $\hat{\beta}$=1.393	0.8	10
$\begin{pmatrix} 0.054 & 0.066 \\ 0.066 & 0.231 \end{pmatrix}$	٠,٢٨٦ ٠,٣٩٣	٠,٣٨٦ ٠,٤٩٥	٠,٣١٧ ٠,٣١٩	$\hat{\alpha}$ =٠,٨١٧ $\hat{\beta}$=١,٤٤٤	0.7	
$\begin{pmatrix} 0.131 & 0.143 \\ 0.143 & 0.295 \end{pmatrix}$	1.754 1.569	0.2322 0.1821	0.754 0.502	$\hat{\alpha}$ =1.254 $\hat{\beta}$=1.627	0.5	
$\begin{pmatrix} 0.02 & 0.018 \\ 0.018 & 0.087 \end{pmatrix}$	0.037 0.105	0.045 0.118	0.09 0.114	$\hat{\alpha}$ =0.59 $\hat{\beta}$=1.239	0.8	20
$\begin{pmatrix} 0.031 & 0.027 \\ 0.027 & 0.095 \end{pmatrix}$	٠,٢٦٧ ٠,١٢٥	٠,٠٨٩ ٠,١٤٦	٠,١٣٣ ٠,١٤	$\hat{\alpha}$ =٠,٦٣٣ $\hat{\beta}$=١,٢٦٥	0.7	
$\begin{pmatrix} 0.091 & 0.064 \\ 0.064 & 0.116 \end{pmatrix}$	0.252 0.163	0.333 0.197	0.283 0.186	$\hat{\alpha}$ =0.783 $\hat{\beta}$=1.311	0.5	

$\begin{pmatrix} 0.013 & 0.011 \\ 0.011 & 0.055 \end{pmatrix}$	0.021 0.063	0.024 0.069	0.06 0.076	$\hat{\alpha}$ =0.56 $\hat{\beta}$ =1.125	0.8	30
$\begin{pmatrix} 0.021 & 0.016 \\ 0.016 & 0.058 \end{pmatrix}$	٠,٠٣٤ ٠,٠٦٧	٠,٠٤ ٠,٠٧٣	٠,٠٨ ٠,٠٨	$\hat{\alpha}$ =٠,٥٨ $\hat{\beta}$ =١,٢٠٥	0.7	
$\begin{pmatrix} 0.063 & 0.039 \\ 0.039 & 0.071 \end{pmatrix}$	0.126 0.09	0.157 0.105	0.176 0.123	$\hat{\alpha}$ =0.676 $\hat{\beta}$ =1.248	0.5	

Table (7): (A.B), Mean Square Error (MSE), variance (VAR) and Sample Information (S.Infor) of the MLE's
Based on type II Censored Sample, at α=1/2, β=9/8 n=10,20,30

S.Infor.	VAR	MSE	Bias_$\hat{\alpha}$, $\hat{\beta}$	Estimator	p	n
$\begin{pmatrix} 8.216 \times 10^{-3} & 6.416 \times 10^{-3} \\ 6.416 \times 10^{-3} & 0.031 \end{pmatrix}$	0.011 0.033	0.012 0.035	0.035 0.044	$\hat{\alpha}$ =0.535 $\hat{\beta}$=1.169	0.8	50
$\begin{pmatrix} 0.013 & 9.112 \times 10^{-3} \\ 9.112 \times 10^{-3} & 0.033 \end{pmatrix}$	٠,٠١٨ ٠,٠٣٥	٠,٠٢ ٠,٠٣٨	٠,٠٤٦ ٠,٠٤٩	$\hat{\alpha}$ =٠,٥٤٦ $\hat{\beta}$=١,١٧٤	0.7	
$\begin{pmatrix} 37.894 & -20.833 \\ -20.833 & 36.893 \end{pmatrix}$	0.057 0.045	0.067 0.05	0.099 0.069	$\hat{\alpha}$ =0.599 $\hat{\beta}$=1.1947	0.5	
$\begin{pmatrix} 4.134 \times 10^{-3} & 3.048 \times 10^{-3} \\ 3.048 \times 10^{-3} & 0.015 \end{pmatrix}$	4.726$\times 10^{-3}$ 0.016	5.02$\times 10^{-3}$ 0.016	0.017 0.024	$\hat{\alpha}$ =0.517 $\hat{\beta}$=1.149	0.8	100
$\begin{pmatrix} 6.503 \times 10^{-3} & 4.328 \times 10^{-3} \\ 4.328 \times 10^{-3} & 0.016 \end{pmatrix}$	٧,٥٧٥$\times 10^{-3}$ ٠,٠١٦	٨,١٠١$\times 10^{-3}$ ٠,٠١٧	٠,٠٢٣ ٠,٠٢٦	$\hat{\alpha}$ =٠,٥٢٣ $\hat{\beta}$=١,١٥١	0.7	
$\begin{pmatrix} 0.019 & 0.01 \\ 0.01 & 0.019 \end{pmatrix}$	٠,٠٢٣ ٠,٠١٩	٠,٠٢٥ ٠,٠٢١	٠,٠٤٧ ٠,٠٣٥	$\hat{\alpha}$ =٠,٥٤٧ $\hat{\beta}$=١,١٦	0.5	

Table (8): (A.B), Mean Square Error (MSE), variance (VAR) and Sample Information (S.Infor) of the MLE's Based on type II Censored Sample, at α=9/8, β=9/8 n=10,20,30

S.Infor	VAR	R.MSE	MSE	R.A.B	Bias_ $\hat{\alpha}$, $\hat{\beta}$	Estimator	p	n
$\begin{pmatrix}0.161 & 0.127\\0.127 & 0.191\end{pmatrix}$	0.449 0.304	0.533 0.335	0.6 0.377	0.345 0.24	0.388 0.27	$\hat{\alpha}$ =1.513 $\hat{\beta}$=1.395	0.8	10
$\begin{pmatrix}0.235 & 0.172\\0.172 & 0.217\end{pmatrix}$	0.801 0.384	0.96 0.433	1.08 0.488	0.47 0.285	0.529 0.321	$\hat{\alpha}$ =1.654 $\hat{\beta}$=1.446	0.7	
$\begin{pmatrix}0.55 & 0.347\\0.347 & 0.306\end{pmatrix}$	3.792 0.826	4.599 0.971	5.174 1.092	1.045 0.459	1.176 0.516	$\hat{\alpha}$ =2.301 $\hat{\beta}$=1.641	0.5	
$\begin{pmatrix}0.08 & 0.054\\0.054 & 0.081\end{pmatrix}$	0.134 0.1	0.146 0.102	0.164 0.115	0.154 0.108	0.173 0.122	$\hat{\alpha}$ =1.298 $\hat{\beta}$=1.247	0.8	20
$\begin{pmatrix}0.118 & 0.072\\0.072 & 0.089\end{pmatrix}$	0.208 0.115	0.231 0.119	0.26 0.134	0.201 0.124	0.226 0.14	$\hat{\alpha}$ =1.351 $\hat{\beta}$=1.265	0.7	
$\begin{pmatrix}0.293 & 0.144\\0.144 & 0.119\end{pmatrix}$	0.649 0.176	0.757 0.193	0.852 0.217	0.4 0.181	0.45 0.203	$\hat{\alpha}$ =1.572 $\hat{\beta}$=1.328	0.5	
$\begin{pmatrix}0.053 & 0.034\\0.034 & 0.051\end{pmatrix}$	0.072 0.058	0.074 0.056	0.083 0.063	0.094 0.068	0.106 0.076	$\hat{\alpha}$ =1.231 $\hat{\beta}$=1.201	0.8	30
$\begin{pmatrix}0.078 & 0.045\\0.045 & 0.056\end{pmatrix}$	0.109 0.064	0.114 0.063	0.128 0.071	0.123 0.077	0.139 0.086	$\hat{\alpha}$ =1.264 $\hat{\beta}$=1.211	0.7	
$\begin{pmatrix}0.191 & 0.087\\0.087 & 0.071\end{pmatrix}$	0.303 0.09	0.332 0.093	0.373 0.104	0.236 0.108	0.265 0.122	$\hat{\alpha}$ =1.39 $\hat{\beta}$=1.247	0.5	

Table (9): (A.B), Mean Square Error (MSE), variance (VAR) and Sample Information (S.Infor) of the MLE's Based on type II Censored Sample, at α=9/8, β=9/8 n=50,100

S.Infor	VAR	R.MSE	MSE	R.A.B	**Bias_** $\hat{\alpha}$, $\hat{\beta}$	Estimator	p	n
$\begin{pmatrix} 0.032 & 0.019 \\ 0.019 & 0.029 \end{pmatrix}$	0.039 0.031	0.039 0.03	0.043 0.034	0.059 0.043	0.066 0.049	$\hat{\alpha}$ =1.191 $\hat{\beta}$=1.174	0.8	50
$\begin{pmatrix} 0.046 & 0.026 \\ 0.026 & 0.032 \end{pmatrix}$	0.057 0.034	0.057 0.033	0.064 0.037	0.076 0.049	0.085 0.055	$\hat{\alpha}$ =1.21 $\hat{\beta}$=1.18	0.7	
$\begin{pmatrix} 0.113 & 0.049 \\ 0.049 & 0.04 \end{pmatrix}$	0.147 0.045	0.152 0.045	0.171 0.05	0.139 0.066	0.156 0.074	$\hat{\alpha}$ =1.281 $\hat{\beta}$=1.199	0.5	
$\begin{pmatrix} 0.016 & 9.281 \times 10^{-3} \\ 9.281 \times 10^{-3} & 0.014 \end{pmatrix}$	0.017 0.015	0.016 0.013	0.018 0.015	0.026 0.018	0.029 0.02	$\hat{\alpha}$ =1.154 $\hat{\beta}$=1.145	0.8	100
$\begin{pmatrix} 0.023 & 0.012 \\ 0.012 & 0.015 \end{pmatrix}$	0.026 0.016	0.024 0.015	0.027 0.017	0.034 0.021	0.038 0.023	$\hat{\alpha}$ =1.163 $\hat{\beta}$=1.148	0.7	
$\begin{pmatrix} 0.055 & 0.023 \\ 0.023 & 0.019 \end{pmatrix}$	0.064 0.02	0.061 0.019	0.069 0.021	0.063 0.028	0.071 0.032	$\hat{\alpha}$ =1.196 $\hat{\beta}$=1.157	0.5	

Table (10): (A.B), Mean Square Error (MSE), variance (VAR) and Sample Information (S.Infor) of the MLE's Based on type II Censored sample at α=9/8, β=1/8 n=10,20,30

S. Infor.	VAR	R.MSE	MSE	R.A.B	$\hat{\beta},\hat{\alpha}$ Bias_	Estimator	p	n
$\begin{pmatrix} 0.163 & 9.181\times10^{-3} \\ 9.181\times10^{-3} & 0.034 \end{pmatrix}$	0.165 0.063	0.175 1.628	0.197 0.203	0.159 2.996	-0.179 0.374	=0.946$\hat{\alpha}$ =0.499$\hat{\beta}$	0.8	10
$\begin{pmatrix} 0.173 & 8.719\times10^{-3} \\ 8.719\times10^{-3} & 0.038 \end{pmatrix}$	0.237 0.088	0.251 1.937	0.283 0.242	0.191 3.136	-0.215 0.392	=0.91$\hat{\alpha}$ =0.517$\hat{\beta}$	0.7	
$\begin{pmatrix} 0.199 & 0.03 \\ 0.03 & 0.115 \end{pmatrix}$	1.2 0.81	1.079 9.48	1.214 1.185	0.104 4.902	-0.117 0.613	=1.008$\hat{\alpha}$ =0.738$\hat{\beta}$	0.5	
$\begin{pmatrix} 0.071 & 2.694\times10^{-3} \\ 2.694\times10^{-3} & 0.014 \end{pmatrix}$	0.047 0.015	0.003 0.923	0.127 0.115	0.251 2.534	-0.283 0.317	=0.842$\hat{\alpha}$ =0.432$\hat{\beta}$	0.8	20
$\begin{pmatrix} 0.072 & 2.376\times10^{-3} \\ 2.376\times10^{-3} & 0.014 \end{pmatrix}$	0.05 0.018	0.14 0.978	0.157 0.122	0.291 2.579	-0.328 0.322	=0.797$\hat{\alpha}$ =0.447$\hat{\beta}$	0.7	
$\begin{pmatrix} 0.072 & 3.401\times10^{-3} \\ 3.401\times10^{-3} & 0.027 \end{pmatrix}$	0.086 0.063	0.185 1.82	0.209 0.008	0.311 3.249	-0.35 0.406	=0.775$\hat{\alpha}$ =0.531$\hat{\beta}$	0.5	

$\begin{pmatrix} 0.045 & 1.59\times10^{-3} \\ 1.59\times10^{-3} & 8.838 \end{pmatrix}$	0.026 × 10^{-3} 8.866	0.108 0.825	0.121 0.103	0.275 2.457	-0.309 0.307	$\hat{\alpha}$=0.816 $\hat{\beta}$=0.432	0.8	30
$\begin{pmatrix} 0.046 & 1.425\times10^{-3} \\ 1.425\times10^{-3} & 9.919\times10^{-3} \end{pmatrix}$	0.025 × 10^{-3} 9.772	0.132 0.849	0.149 0.106	0.313 2.483	-0.352 0.31	$\hat{\alpha}$=0.773 $\hat{\beta}$=0.435	0.7	
$\begin{pmatrix} 0.045 & 1.443\times10^{-3} \\ 1.443\times10^{-3} & 0.016 \end{pmatrix}$	0.039 0.037	0.17 1.463	0.191 0.183	0.347 3.058	-0.39 0.382	$\hat{\alpha}$=0.735 $\hat{\beta}$=0.507	0.5	

Table (11): (A.B), Mean Square Error (MSE), variance (VAR) and Sample Information (S.Infor) of the MLE's Based on type II Censored Sample, at α=9/8, β=1/8 n=50,100

S. Infor.	VAR	R.MSE	MSE	R.A.B	Bias_ $\hat{\beta}$,$\hat{\alpha}$	Estimator	p	n
$\begin{pmatrix} 0.026 & 8.251\times10^{-4} \\ 8.251\times10^{-4} & 5.081\times10^{-3} \end{pmatrix}$	0.013 × 10^{-3} 4.48	0.11 0.738	0.124 0.092	0.296 2.37	-0.333 0.296	$\hat{\alpha}$=0.792 $\hat{\beta}$=0.421	0.8	50
$\begin{pmatrix} 0.026 & 7.541\times10^{-4} \\ 7.541\times10^{-4} & 5.248\times10^{-3} \end{pmatrix}$	0.012 × 10^{-3} 4.716	0.135 0.751	0.152 0.094	0.332 2.388	-0.374 0.298	$\hat{\alpha}$=0.751 $\hat{\beta}$=0.423	0.7	

$\begin{pmatrix} 0.026 & 5.606\times10^{-4} \\ 5.606\times10^{-4} & 8.334\times10^{-3} \end{pmatrix}$	0.015 0.025	0.165 1.229	0.185 0.154	0.367 2.872	-0.413 0.359	$=0.712\hat{\alpha}$ $=0.484\hat{\beta}$	0.5	
$\begin{pmatrix} 0.013 & 3.749\times10^{-4} \\ 3.749\times10^{-4} & 2.47\times10^{-3} \end{pmatrix}$	$\times 10^{-3}$ 6.279 $\times 10^{-3}$ 1.927	0.114 0.683	0.128 0.085	0.31 2.312	-0.349 0.414	$=0.776\hat{\alpha}$ $=0.414\hat{\beta}$	0.8	100
$\begin{pmatrix} 0.013 & 3.503\times10^{-4} \\ 3.503\times10^{-4} & 2.541\times10^{-3} \end{pmatrix}$	$\times 10^{-3}$ 5.627 $\times 10^{-3}$ 1.998	0.139 0.692	0.156 0.086	0.345 2.325	-0.388 0.291	$=0.737\hat{\alpha}$ $=0.416\hat{\beta}$	0.7	
$\begin{pmatrix} 0.013 & 1.967\times10^{-4} \\ 1.967\times10^{-4} & 3.596\times10^{-4} \end{pmatrix}$	$\times 10^{-3}$ 5.172 $\times 10^{-3}$ 6.56	0.167 0.922	0.188 0.115	0.38 2.638	-0.427 0.33	$=0.698\hat{\alpha}$ $=0.455\hat{\beta}$	0.5	

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