# Recurrence Relations for Single and Product Moments of *k*-th Record Values from Modified Weibull Distribution and a Characterization

**Dr. Mahmoud Ali Selim**

Department of Statistics, Faculty of commerce,

Al Azhar University, Cairo, Egypt

selim.one@gmail.com

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selim.one@gmail.com

## Abstract

In this paper, some recurrence relations satisfied by single and product moments of *k-th* upper record values from the modified Weibull distribution are established. Further, a characterization of the modified Weibull distribution based on recurrence relation for single moments of *k-th* upper record values is presented.

**Mathematics Subject Classification:** 62G30, 62G99, 62E10

**Keywords:** Single moments, product moments, *k-th* upper record values, recurrence relations, modified Weibull distribution, characterization

## 1 Introduction

A new life time distribution named a modified Weibull distribution is recently proposed by Sarhan and Zaindin (2009) as a generalization for some most commonly used distributions in reliability and life testing, such as exponential, Rayleigh, linear failure rate and Weibull distribution. These distributions have several desirable properties and nice physical interpretations. So, the modified Weibull distribution that generalizes all the above distributions can be widely and effectively used in reliability applications because it has a wide variety of shapes in its density and hazard functions. These properties and more have been studied by Sarhan and Zaindin (2009).

A random variable $X$ is said to have the modified Weibull distribution if its probability density function (pdf) and cumulative distribution function (cdf) given respectively by

$$f(x) = (\alpha + \beta\gamma x^{\gamma-1})e^{-\alpha x - \beta x^{\gamma}}, \;\; x > 0, \;\; \gamma > 0, \;\; \alpha, \beta \geq 0, \qquad (1.1)$$

and

$$F(x) = 1 - e^{-\alpha x - \beta x^{\gamma}}, \;\; x > 0, \;\; \gamma > 0, \;\; \alpha, \beta \geq 0 \qquad (1.2)$$

This distribution reduces to the exponential distribution ED($\alpha$) when $\beta = 0$, Raleigh distribution RD($\beta$) when $\alpha = 0, \gamma = 2$, Linear failure rate distribution LFRD$(\alpha, \beta)$ when $\gamma = 2$ and Weibull distribution WD$(\beta, \gamma)$ when $\alpha = 0$. More details on this distribution can be found in Sarhan and Zaindin (2009), Zaindin and Sarhan (2009), Al-Hadhrami (2010), Zaindin (2010) and Gasmi and Berzig (2011). Notation of MWD ($\alpha,\beta,\gamma$) is used to denote the modified Weibull distribution with three parameters $\alpha, \beta$ and $\gamma$ in form (1.1).

Record values arise naturally in several real-life problems including industrial stress testing, meteorological analysis, hydrology, athletic events and mining surveys. The formal study of record values theory started with Chandler (1952). Theory of record values and its distributional properties have been extensively studied in the literature, for example, see, Ahsanullah (1995), Nagaraga (1988), Arnold et. al. (1998) and Selim (2012).

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a cumulative distribution function $F(x)$ and probability density function $f(x)$. An observation $X_j$ will be called an upper record value if its value exceeds all previous observations. Thus, $X_j$ is an upper record value if $X_j > X_i$ for every $i < j$. For a fixed positive integer $k$, Dziubdziela and Kopocinski (1976) defined the sequence $\left\{U_n^{(k)}, n \geq 1\right\}$ of *k-th* upper record times for the sequence $\{X_n, n \geq 1\}$ as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = min\left\{j > U_n^{(k)}: X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1}\right\},$$

where $X_{j:m}$ is the *j-th* order statistic of the sample $X_1, X_2, \dots, X_m$. Then the sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called a sequence of *k-th* upper record values of $\{X_n, n \geq 1\}$. Note that for $k$ = 1, we get the usual upper record values as defined in Chandler (1952). For convenience, we also take $Y_0^{(k)} = 0$ and $Y_1^{(k)} = min(X_1, X_2, \dots, X_n) = X_{1:n}$.

The moments of k-records have received considerable attention in the recent years. Many authors have been obtained several recurrence relations of single as well as product moments of k-records for different distributions. Also some of them studied the problems of characterizing the probability distributions based on the moments of k-records, for example see, Grudzien and Szynal (1997), Pawlas and Szynal (1998, 1999, 2000), Saran and Singh (2008), Bieniek and Szynal (2002, 2007), Shawky (2008), Nain (2010) and Kumar (2011).

The recurrence relations for moments of *k*-records from modified Weibull distribution in (1.1) did not derive in the available literature. Hence, in this paper, we established some recurrence relations for the single and product moments of *k-th* upper record values from the modified Weibull distribution in (1.1). Further, its various deductions and particular cases are discussed. Also, a characterization of this distribution has been obtained on using a recurrence relation for single moments.

We shall denote

$$\mu_{n:k}^{(r)} = E\left[\left(Y_n^{(k)}\right)^r\right], \quad r, n = 1,2, \dots,$$
$$\mu_{m,n:k}^{(r,s)} = E\left[\left(Y_m^{(k)}\right)^r \left(Y_n^{(k)}\right)^s\right], 1 \leq m \leq n-1, \ \ r,s = 0,1,2, \dots,$$
$$\mu_{m,n:k}^{(r,0)} = E\left[\left(Y_m^{(k)}\right)^r \left(Y_n^{(k)}\right)^0\right] = \mu_{m:k}^{(r)}, 1 \leq m \leq n-1, r,s = 0,1,2, \dots,$$
$$\mu_{m,n:k}^{(0,s)} = E\left[\left(Y_m^{(k)}\right)^0 \left(Y_n^{(k)}\right)^s\right] = \mu_{n:k}^{(s)}, 1 \leq m \leq n-1, \ \ r,s = 0,1,2, \dots .$$

## 2 Recurrence Relations for Single and Product Moments

Now, we can note from (1.1) and (1.2) that for the modified Weibull distribution

$$f(x) = \left[\frac{\gamma}{x}(-\ln \bar{F}(x)) - \alpha(\gamma - 1)\right] \bar{F}(x) \tag{2.1}$$

where $\bar{F}(x) = 1 - F(x)$.

The relation in (2.1) will be used in this paper to derive some recurrence relations for the single and product moments of *k-th* upper record values from MWD($\alpha,\beta,\gamma$).

Let $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ be a sequence of *k-th* upper record values arising from MWD($\alpha,\beta,\gamma$) in (1.1). Then the probability density function of $Y_n^{(k)}$, $n \geq 1$, (see Dziubdziela and Kopocinski, (1976)) is

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}\,[-\ln \bar{F}(x)]^{n-1}\,[\bar{F}(x)]^{k-1} f(x), \quad -\infty < x < \infty \quad (2.2)$$

and the joint density function of $Y_m^{(k)}$ and $Y_n^{(k)}$, $1 \leq m < n,\ n \geq 2,$ is

$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y) = \frac{k^n}{(m-1)!\,(n-m-1)!}\,[-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-1} \\ \times [-\ln \bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[\bar{F}(y)]^{k-1} f(y), \quad x < y \quad (2.3)$$

Now, we can introduce the following recurrence relations.

**Theorem 1**

For $n \geq 2$ and $r = 1, 2, \ldots,$

$$\mu_{n:k}^{(r)} = \frac{n\gamma}{r}\left(\mu_{n+1:k}^{(r)} - \mu_{n:k}^{(r)}\right) - \frac{k\alpha\,(\gamma - 1)}{(r+1)}\left(\mu_{n:k}^{(r+1)} - \mu_{n-1:k}^{(r+1)}\right) \quad (2.4)$$

and for n = 1

$$\mu_{1:k}^{(r)} = \frac{\gamma}{r}\left(\mu_{2:k}^{(r)} - \mu_{1:k}^{(r)}\right) - \frac{k\alpha(\gamma - 1)}{(r+1)}\left(\mu_{1:k}^{(r+1)}\right). \quad (2.5)$$

**Proof**

For $n \geq 2$, and $r = 0, 1, 2, \ldots$. Using the pdf of $X_{U(n)}$ given in (2.2) and the relation in (2.1), we have

$$\mu_{n:k}^{(r)} = \frac{k^n}{(n-1)!}\left\{\begin{array}{c}\gamma\int_0^\infty x^{r-1}\,[-\ln \bar{F}(x)]^n[\bar{F}(x)]^k dx \\ -\alpha(\gamma-1)\int_0^\infty x^r\,[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx\end{array}\right\} \qquad (2.6)$$

Integrating by parts, in the first term treating $x^{r-1}$ for integration and the rest of the integrand for differentiation and in the second term treating $x^r$ for integration and the rest of the integrand for differentiation, we get

$$\mu_{n:k}^{(r)} = \frac{\gamma k^n}{(n-1)!\ r}\left\{\begin{array}{c}k\int_0^\infty x^r\,[-\ln \bar{F}(x)]^n[\bar{F}(x)]^{k-1}f(x)dx \\ -n\int_0^\infty x^r\,[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx\end{array}\right\}$$
$$-\frac{\alpha(\gamma-1)k^n}{(n-1)!\ (r+1)}\left\{\begin{array}{c}k\int_0^\infty x^{r+1}\,[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx \\ -(n-1)\int_0^\infty x^{r+1}\,[-\ln \bar{F}(x)]^{n-2}[\bar{F}(x)]^{k-1}f(x)dx\end{array}\right\} \qquad (2.7)$$

Upon rewriting the above expression, we immediately obtain the recurrence relation (2.4). Moreover, the relation (2.5) follows from (2.7) simply by setting n=1.

**Remark 1**

By putting $k = 1$, in (2.4) and (2.5) we can deduce the recurrence relations for single moments of the usual upper record values from the modified Weibull distribution.

**Special cases**

(a) Setting $\alpha = 0$, in (2.4) and simplifying, we get the recurrence relation for single moments of *k-th* upper record values from weibull distribution as follow

$$\mu_{n+1:k}^{(r)} = \mu_{n:k}^{(r)}\left[1+\frac{r}{n\gamma}\right], \qquad n \geq 1.$$

(b) Setting $\alpha = 0, \gamma = 2$, in (2.4) and simplifying, we get the recurrence relation for single moments of *k-th* upper record values from Rayleigh distribution as follow

$$\mu_{n+1:k}^{(r)} = \mu_{n:k}^{(r)}\left[1 + \frac{r}{2n}\right], \qquad n \geq 1.$$

(c) Setting $\alpha = 0, \gamma = 1$, in (2.4) and simplifying, we get the recurrence relation for single moments of *k-th* upper record values from exponential distribution as follow

$$\mu_{n+1:k}^{(r)} = \mu_{n:k}^{(r)}\left[1 + \frac{r}{n}\right], \qquad n \geq 1.$$

(d) Setting $\gamma = 2$, in (2.4), we get the recurrence relation for single moments of *k-th* upper record values from linear failure rate distribution as follow

$$\mu_{n:k}^{(r+1)} = \mu_{n-1:k}^{(r+1)} + \frac{2n(r+1)}{k\ r\ \alpha}\left[\mu_{n+1:k}^{(r)} - \mu_{n:k}^{(r)}\left(1 + \frac{r}{2n}\right)\right], \qquad n > 1$$

and for $n = 1$

$$\mu_{1:k}^{(r+1)} = \frac{2(r+1)}{k\ r\ \alpha}\left[\mu_{2:k}^{(r)} - \mu_{1:k}^{(r)}\left(1 + \frac{r}{2}\right)\right].$$

**Theorem 2**

For $1 \leq m \leq n-2$ and $r, s = 0,1,2, \ldots,$

$$\mu_{m,n:k}^{(r,s)} = \frac{m\gamma}{r}\left(\mu_{m+1,n:k}^{(r,s)} - \mu_{m,n:k}^{(r,s)}\right) - \frac{k\alpha\,(\gamma-1)}{(r+1)}\left(\mu_{m-1,n-1:k}^{(r+1,s)} - \mu_{m,n-1:k}^{(r+1,s)}\right) \tag{2.8}$$

and for $m \geq 1$ and $r, s = 0,1,2, \ldots,$

$$\mu_{m,m+1:k}^{(r,s)} = \frac{m\gamma}{r}\left(\mu_{m+1:k}^{(r+s)} - \mu_{m,m+1:k}^{(r,s)}\right) - \frac{k\alpha\,(\gamma-1)}{(r+1)}\left(\mu_{m-1,m:k}^{(r+1,s)} - \mu_{m:k}^{(r+s+1)}\right) \tag{2.9}$$

**Proof**

From (2.3) and (2.1), for $1 \le m \le n-1,$ and $r, s = 0,1,2, \ldots$ we have

$$\mu_{m,n:k}^{(r,s)} = \frac{k^n}{(m-1)!\,(n-m-1)!} \int_0^{\infty} y^s\, I(y)\, [\bar{F}(y)]^{k-1} f(y) dy \qquad (2.10)$$

where

$$I(y) = \gamma \int_0^{y} x^{r-1}\, [-\ln \bar{F}(x)]^{m} [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-1} dx$$
$$-\alpha(\gamma-1) \int_0^{y} x^{r}\, [-\ln \bar{F}(x)]^{m-1} [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-1} dx$$

Integrating $I(y)$ by parts, in the first term treating $x^{r-1}$ for integration and the rest of the integrand for differentiation and in the second term treating $x^r$ for integration and the rest of the integrand for differentiation, we get

$$I(y) = \frac{\gamma}{r} \begin{Bmatrix} (n-m-1) \int_0^{y} x^{r}\, [-\ln \bar{F}(x)]^{m} [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-2} \frac{f(x)}{\bar{F}(x)} dx \\ -m \int_0^{y} x^{r}\, [-\ln \bar{F}(x)]^{m-1} [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-1} \frac{f(x)}{\bar{F}(x)} dx \end{Bmatrix}$$
$$- \frac{\alpha(\gamma-1)}{(r+1)} \begin{Bmatrix} (n-m-1) \int_0^{y} x^{r+1}\, [-\ln \bar{F}(x)]^{m-1} [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-2} \frac{f(x)}{\bar{F}(x)} dx \\ -(m-1) \int_0^{y} x^{r+1}\, [-\ln \bar{F}(x)]^{m-2} [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-1} \frac{f(x)}{\bar{F}(x)} dx \end{Bmatrix}$$

Upon substituting the above expression in (2.10) and simplifying, we obtain (2.8). And when $n = m+1$, then

$$\mu_{m,n:k}^{(r,s)} = \frac{k^n}{(m-1)!} \int_0^{\infty} y^s\, I(y)\, [\bar{F}(y)]^{k-1} f(y) dy \qquad (2.11)$$

where

$$I(y) = \gamma \int_0^{y} x^{r-1}\, [-\ln \bar{F}(x)]^{m} dx \; - \alpha(\gamma-1) \int_0^{y} x^{r}\, [-\ln \bar{F}(x)]^{m-1} dx$$

Integrating $I(y)$ by parts, we get

$$I(y)=\frac{\gamma}{r}\left\{y^{r}[-\ln\bar{F}(y)]^{m}-m\int_{0}^{y}x^{r}\,[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}dx\right\}$$
$$-\frac{\alpha(\gamma-1)}{(r+1)}\left\{y^{r+1}[-\ln\bar{F}(y)]^{m-1}+(m-1)\int_{0}^{y}x^{r+1}\,[-\ln\bar{F}(x)]^{m-2}\frac{f(x)}{\bar{F}(x)}dx\right\}$$

Substituting the above expression in (2.11) and simplifying, we obtain (2.9).

**Remark 2**

Putting $k$ = 1, in (2.8) and (2.9) we deduce the recurrence relations for the product moments of usual upper record values from the modified Weibull distribution.

**Remark 3**

In Theorem 2 if we put

(a) $\alpha=0$, we get results for Weibull distribution.
(b) $\alpha=0,\gamma=2$, we get results for Rayleigh distribution.
(c) $\alpha=0,\gamma=1$, we get results for exponential distribution.
(d) $\gamma=2$, we get results for linear failure rate distribution.

## 3 A Characterization of the Modified Weibull Distribution

In this section, we present a characterization for modified Weibull distribution using the relation in (2.4) based on the following result of Lin (1986).

**Proposition 1**

Let $n_0$ be any fixed non-negative integer and let $a,b$ be real numbers such that $-\infty<a<b<\infty$. Let $g(x)\geq 0$ be an absolutely continuous function with $g'(x)\neq 0$ almost everywhere on $(a,b)$. Then the sequence of functions $\left\{[g(x)]^{n}e^{-g(x)},n\geq n_0\right\}$ is complete in $L(a,b)$ if and only if $g(x)$ is strictly monotone on $(a,b)$.

**Theorem 3**

For a fixed positive integer $k$. A necessary and sufficient condition for a random variable $X$ to be distributed with pdf given by (1.1) is that

$$\mu_{n:k}^{(r)} = \frac{n\gamma}{r}\left(\mu_{n+1:k}^{(r)} - \mu_{n:k}^{(r)}\right) - \frac{k\alpha\,(\gamma-1)}{(r+1)}\left(\mu_{n:k}^{(r+1)} - \mu_{n-1:k}^{(r+1)}\right) \tag{3.1}$$

**Proof**

The necessary part follows immediately from (2.4) on the other hand if the recurrence relation in (3.1) is satisfied, then on rearranging the terms in (3.1) and using (2.2), we have

$$\begin{aligned}
-\frac{\alpha\,(\gamma-1)k^{n+1}}{(n-1)!\,(r+1)}\int_0^\infty x^{r+1}\,[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx \\
= \frac{k^n}{(n-1)!}\int_0^\infty x^r\,[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx \\
- \frac{\gamma k^{n+1}}{(n-1)!\,r}\int_0^\infty x^r\,[-\ln\bar{F}(x)]^{n}[\bar{F}(x)]^{k-1}f(x)dx \\
+ \frac{\gamma k^{n}}{(n-2)!\,r}\int_0^\infty x^r\,[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx \\
- \frac{\alpha\,(\gamma-1)k^{n}}{(n-1)!\,(r+1)}\int_0^\infty x^{r+1}\,[-\ln\bar{F}(x)]^{n-2}[\bar{F}(x)]^{k-1}f(x)dx
\end{aligned} \tag{3.2}$$

Integrating the last two integrals on the RHS of (3.2) by parts, we get

$$\begin{aligned}
-\frac{\alpha\,(\gamma-1)k^{n+1}}{(n-1)!\,(r+1)}\int_0^\infty x^{r+1}\,[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx \\
= \frac{k^n}{(n-1)!}\int_0^\infty x^r\,[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx \\
- \frac{\gamma k^{n+1}}{(n-1)!\,r}\int_0^\infty x^r\,[-\ln\bar{F}(x)]^{n}[\bar{F}(x)]^{k-1}f(x)dx \\
- \frac{\gamma k^{n}}{(n-1)!}\int_0^\infty x^{r-1}\,[-\ln\bar{F}(x)]^{n}[\bar{F}(x)]^{k}dx \\
+ \frac{\gamma k^{n+1}}{(n-1)!\,r}\int_0^\infty x^r\,[-\ln\bar{F}(x)]^{n}[\bar{F}(x)]^{k-1}f(x)dx \\
+ \frac{\alpha\,(\gamma-1)k^{n}}{(n-1)!}\int_0^\infty x^r\,[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k}dx \\
- \frac{\alpha\,(\gamma-1)k^{n+1}}{(n-1)!\,(r+1)}\int_0^\infty x^{r+1}\,[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx
\end{aligned}$$

which after simplification reduces to

$$\frac{k^n}{(n-1)!}\int_0^\infty x^r\,[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}$$
$$\times\left\{f(x)-\left[\frac{\gamma}{x}(-\ln \bar{F}(x))-\alpha\,(\gamma-1)\right]\bar{F}(x)\right\}dx=0 \quad (3.3)$$

Now follows from proposition 1 that

$$f(x)=\left[\tfrac{\gamma}{x}(-\ln \bar{F}(x))-\alpha\,(\gamma-1)\right]\bar{F}(x)\,.$$

This proves by (2.1), that $f(x)$ is pdf of modified Weibull distribution in the form (1.1).

## Concluding Remarks

(1) In this paper, some recurrence relations for single and product moments of *k-th* upper record values from the modified Weibull distribution have been derived.
(2) Several recurrence relations for single and product moments of *k-th* upper record values have been deduced for exponential, Rayleigh, linear failure rate and Weibull distributions as special cases.
(3) The recurrence relation for single moments of *k-th* upper record values has been utilized to obtain a characterization of the modified Weibull distribution.
(4) The recurrence relations for moments of *k*-records are important for the following reasons:
   (a) They provide general results that can be applied for their special cases.
   (b) They help in reducing the amount of computational complexities.
   (c) They can be used in a simple recursive manner to compute all the single and product moments of *k-th* record values for any sample size.
   (d) They can be used to characterize distributions.

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