Rayleigh Uniform {Log-logistic} distribution And Aplications

Abdelhamid M. Rabie[(1)], Mostafa Abdelhamid[(2)] and Mahmoud Wahba[(3)].

Abstract: In this paper we introduce Rayleigh uniform {log-logsitic} (RU {LL}) distribution. Properties of RU {LL} distribution namely, density function g(x), the ordinary moments, quintile function, mean residual life, Renyi entropy are introduced. Four methods of estimation of the RU{LL} distribution based on complete sampling are introduced. A Monte Carlo simulation study based on R software to evaluate the performance of the estimation methods and to calculate the measures and main formulas introduced in this paper.

Keywords: Rayleigh distribution, $T-\chi$ families, mean residual life, Renyi entropy.

1- Introduction

Statistical distributions have been extensively used over the past decades for modeling many real world important phenomena in several areas such as engineering, medical sciences, biological studies, economics, finance and lifetime analysis. Many new methods have been developed to generate statistical distributions. These methods are based on the idea of combining two or more existing distributions or adding extra parameters to an existing distribution to generate a new family of distributions.

A common feature of these generalized distributions is that they have more parameters. A brief summary of some methods in the literature that related to this article is introduced below.

The person system of continuous distributions as developed by person [26], is a system for which every PDF function $f(x)$ satisfies a certain differential equation. The different shapes of the person distributions were classified by person into a number of types. For details of the various types, see chapter 12 of Johnson et al. [11]. Burr [6] proposed a system of distributions satisfy a certain differential equation. Burr [6] gave 12 types of distributions. See fry [9] and Jonson et al. [11] for a list of Burr types of distributions. Jonson [10] proposed a system for generating distributions using a general normalization transformation. He propose some families of distributions which cover many commonly used distributions such as normal, lognormal gamma, beta and others. For more details see Johnson et al. [11]. Azzalin [3] introduced the skew normal family of distributions. For review of skew-symmetric

([1]) Professor of Statistics, Faculty of Commerce Alazhar University.

([2]) Graduate Student, Faculty of Graduated Studies for Statistical Research, Cairo University.

([3]) Assistant Teacher, Faculty of Commerce Alazhar University.

distributions see Kotz and Vicari (2005) [16]. Marshall and Olkin (1997)[20] proposed a general method for generating a new family of life distributions determined in terms of survival functions. For details about life distributions, one may refer to Marshall and Olkin (2010)[21] and Lai [19]. Eugene et. al. (2002) [8] used the beta distribution as a generator to develop a new family of beta-generated family of distributions, where beta distribution with PDF b(.) is used as the generator. The CDF function of a beta – generated random variable X is defined as:

$$G(x) = \int_0^{F(x)} b(t)dt \ldots\ldots \tag{1.1}$$

Where b(t) is the PDF of the beta random variable and $F(x)$ is the CDF of any random variable. If X is continuous, the PDF corresponding to (1.1) is given by:

$$g(x) = \frac{1}{B(\propto, B)} f(x)F(x)^{\alpha-1} (1 - F(x))^{B-1} \tag{1.2}$$

Where $B(\alpha, \beta)$ is the complete beta function. Since Eugene et al. [8] many beta – generated distributions have been introduced in the literature including the beta Gumbel distribution by Nadarajah and Kotz [24], beta- exponential distribution by Najarah and Kotz [25], and others. Recently Jones [12] and Cordeiro and De Castro [7] extended the beta – generated family by replacing the beta distribution in (1.1) with the Kumaraswamy distribution [18]. Alzaatreh et.al. (2013b)[2] proposed a general method to generate families of distributions with a PDF r(.) of continuous random variable and applying a function $W(F(x))$ that satisfies some conditions to develop the T-X family. The CDF of the T-X family is defined as:

$$G(x) = \int_a^{w(F(x))} r(t)dt \tag{1.3}$$

$$= R\,\{W(F(x)\}$$

Where F(x) is the CDF of any existing distribution with PDF $f(x) and\ R(.)$ is the CDF of T. the corresponding PDF of the T-X family is:

$$g(x) = \left\{\frac{d}{dx} W(F(x))\right\} r\,\{W(F(x))\} \tag{1.4}$$

Different W functions generate different families of T- X distribution.

2- Generating Families of Continuous Probability Distribution Using Quantile Function.

Aljarrah et. al. (2014)[1] generate families of continuous probability distributions using quintile function. They introduced a class of W(.) functions wider than that of Alzatteh et. al. (2013b)[2]. They introduced a general definition of the W(.) function which proposes a general method to generate T-X families.

If T has PDF r(.) with support (a, b), then:

$$G(x) = \int_a^{W(F(x))} r(t)dt \tag{2.1}$$

If both functions W and F are absolutely continuous, then G(x) in (2.1) is absolutely continuous and has a PDF $g(x) = \frac{d}{dx} G(x)$. Let P(.) be the CDF of a random variable y taking values on (a, b) and the quantile function of y denoted by $Q_y(\lambda), \lambda \in (0,1)$. If P(.) is continuous and strictly increasing then $Q_y(\lambda) = P^{-1}(\lambda)$ is continuous and strictly increasing. Taking W(.) to be the quantile function of strictly increasing distribution function P(.) for the random variable y, namely, $W(\lambda) = Q_y(\lambda), \lambda \in (0,1)$, then $Q_y(.)$ is continuous and non-decreasing, and the CDF of a $T - X\{y\}$ family using the quintile function $Q_y(.)$ is defined as:

$$G(x) = \left.\begin{aligned} &\int_a^{Q_y(F(x))} r(t)dt \\ &= R\left\{Q_y(F(x))\right\}, x \in (-\infty, \infty) \end{aligned}\right\} \tag{2.2}$$

Where R(.) is the CDF of T.

If we assume further that y has a density $P(y) > 0$ for all y in a neighborhood of $Q_y(\lambda), \lambda \epsilon (0,1)$, then $\frac{d}{d\lambda} Q(\lambda)$ exists and equals $\left[P(Q_y(\lambda)\right]^{-1}$ and hence the corresponding PDF associated with (2.2) is:

$$g(x) = \frac{r\left\{Q_y(F(x)\right\} f(x)}{P\left\{Q_y(F(x))\right\}} \tag{2.3}$$

The PDF defined in (2.3) can be easily used to generate a $T - X\{y\}$ family of distributions by applying the quintile function of any existing distribution.

3- Some $T - X\{y\}$ Families

3.1 Some $T - X\{y\}$ Families Based on Different Quantile Functions:

Let the random variable y follows the long-logistic distribution with parameters α and β. Then the PDF and the quantile functions are respectively

$$p(y) = \frac{(^{\beta}/_{\alpha})(^{y}/_{\alpha})^{\beta-1}}{(1 + (^{y}/_{\alpha})^{\beta})^2}, y \geq 0$$

$$Q_y(\lambda) = \alpha\left(\frac{\lambda}{1-\lambda}\right)^{1/\beta}, \lambda \epsilon (0,1)$$

Therefore,

$$p\left(Q_y(\lambda)\right) = \left({}^{\beta}/_{\alpha}\right)\lambda^{(B-1)/\beta}(1-\lambda)^{(B+1)/\beta},$$

and the definition (2.3) gives the PDF of $T - X\ \{log - logistic\}$ family as:

$$g(x) = \frac{(\alpha/\beta)f(x)r\left\{\alpha\left(\frac{F(x)}{1-F(x)}\right)^{1/\beta}\right\}}{(F(x))^{(B-1)/\beta}\,(1-F(x))^{(B+1)/\beta}} \qquad (3.1)$$

Also, the definition (2.2) gives the CDF of $T - X\ \{log - logistic\}$ family as:

$$G(x) = R\ \{Q\ (F(x))\}, -\infty\ < x\ <\ \infty$$

Then

$$G(x) = R\left\{\alpha\left(\frac{F(x)}{1-F(x)}\right)^{1/\beta}\right\} \qquad (3.2)$$

When $\alpha = \beta = 1$, the family (3.1) reduces to

$$g(x) = \frac{f(x)}{(1-F(x))^2}\ r\left\{\frac{F(x)}{1-F(x)}\right\} \qquad (3.3)$$

And the family (3.2) reduces to

$$G(x) = R\left\{\frac{F(x)}{1-F(x)}\right\} \qquad (3.4)$$

3.2- A New Distribution Derived from T-X{y} Family: Rayleigh-Uniform {log-logistic} Distribution

Let r(.) and R(.) be the PDF and the CDF of the Rayleigh distribution, f(.) and F(.) are the PDF and CDF of uniform distribution, where

$$r(t) = \frac{t}{\sigma^2}\ e^{-t^2/2\sigma^2} \quad ,t > 0\ ,\sigma\ > 0 \qquad (3.5)$$

$$R(t) = 1 - e^{-t^2/2\sigma^2} \quad ,t > 0\ ,\sigma\ > 0 \qquad (3.6)$$

$$f(x) = \frac{1}{b-a} \quad ,a \le x\ \le b \qquad (3.7)$$

$$F(x) = \frac{x-a}{b-a} \quad ,a \le x\ \le b \qquad (3.8)$$

Then, substituting for r(.), R(.), f(.) and F(.) in (3.3) and (3.4) we obtain the PDF g(x) and the CDF G(x) of the Rayleigh-uniform {log-logistic} (RU$\{LL\}$) distribution as:

$$g(x) = \frac{(b-a)}{\sigma^2\,(b-x)^2}\left(\frac{x-a}{b-x}\right)\,e^{-\left(\frac{x-a}{b-x}\right)^2/2\sigma^2} \tag{3.9}$$

Similarly, substituting for $\frac{F(x)}{1-F(x)} = \frac{x-a}{b-x}$, and using the CDF R(.) given in (3.6) we get from (3.4) the CDF of the Rayleigh-uniform {log-logistic} distribution as:

$$G(x) = 1 - \exp\left\{-\left(\frac{x-a}{b-x}\right)^2/2\sigma^2\right\} \tag{3.10}$$

$$a < x < b \qquad , \qquad \sigma > 0$$

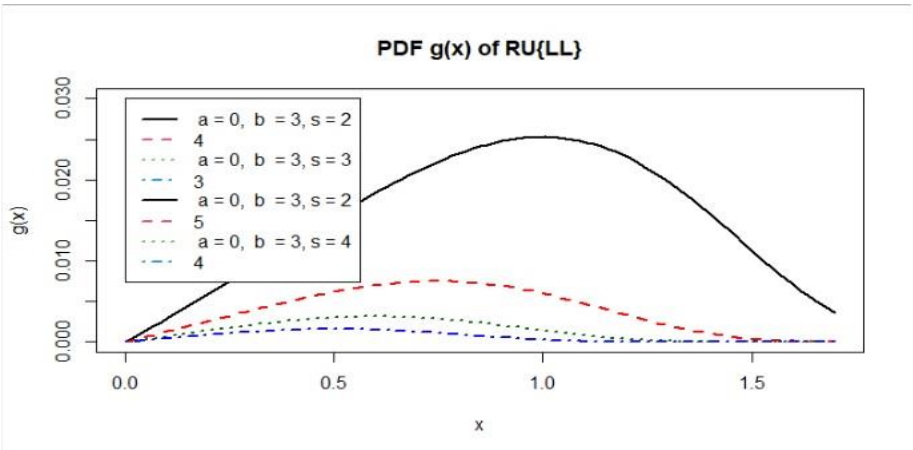
Our new family is called: Rayleigh – uniform {log-logistic} family of T-X{Y} family. The new PDF is g(x) as given in (3.9) and its CDF function is G(x) as given in (3.10)

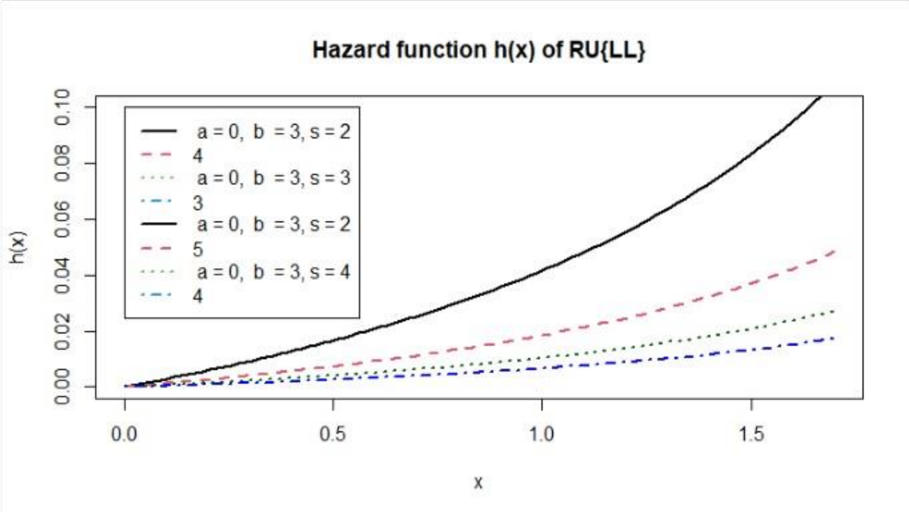
Also the survival function S(x) and the hazard function h(x) are

$$S(x) = 1 - G(x) \tag{3.11}$$

$$h(x) = \frac{g(x)}{S(x)} \tag{3.12}$$

where g(x) and G(x) are as given in (3.9) and (3.10) respectively. g(x) and G(x) will be used in the parameter's estimation. Our parameters are: a , b and σ (3 parameters). The graphs of PDF and hazard functions of RU{LL} are given below





4- The Quantile and Generating Functions

4.1- The Quantile Function

Quantile function plays a significant role in estimation and simulating. In heavy tailed distributions, the measures of skewness and kurtosis based on quantiles are better than those based on moments for which the higher moments may not be exist. The quantile function of RU{LL} distribution can be obtained using the inverse distribution function method as a solution of the equation $G(Q\ (P)) = P$

Then

$$Q_p = \frac{a + b\,\sqrt{2}\,\sigma\,\left[\ln\left(\frac{1}{1-p}\right)\right]^{1/2}}{1 + \sqrt{2}\,\sigma\,\left[\ln\left(\frac{1}{1-p}\right)\right]^{1/2}}\ ,P \in (0,1) \qquad (4.1)$$

The Bowley Skewness measure Bsk [15] and the Moor's Kurtosis measure Mkur [22] are defined by

$$Bsk = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}$$

$$M\ kur = \frac{Q_{0.875} - 2Q_{0.625} - 2Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}$$

The above measures are less sensitive to outliers. Table (1) drawn below gives the values of Bowley Skewness and Moor kurtosis for fixed value of a=0, b=3 and σ = 0.5, 1.5, 2, 2.5, 3, 3.5, 4

Table (1): quantiles, B skewness (Bsk) and M kurtosis (Mkur) for a=0 , b=3 and different values for σ.

Q \ σ	0.5	1.5	2	2.5	3	3.5	4
$Q_{0.25}$	0.82492662	1.5966826	1.8081310	1.9642025	2.0841325	2.179172	2.2563416
$Q_{0.5}$	1.11166958	1.9154464	2.1057643	2.2392593	2.3380742	2.414169	2.4745728
$Q_{0.75}$	1.36294101	2.1422846	2.3071943	2.4189169	2.4996103	2.560625	2.6083774
$Q_{0.125}$	0.61600251	1.3100227	1.5247563	1.6910726	1.8236879	1.931903	2.0218852
$Q_{0.375}$	0.97948484	1.7776603	1.9792722	2.1237932	2.2324655	2.317156	2.3850139
$Q_{0.625}$	1.23560143	2.0325353	2.2107723	2.3335526	2.4232740	2.491704	2.5456176
$Q_{0.875}$	1.51460661	2.2609019	2.4092939	2.5080624	2.5785334	2.631344	2.6723938
Bsk	-0.06593046	-0.1684847	-0.1927675	-0.2098002	-0.2224080	-0.232117	-0.2398239
Mkur	-0.15701653	-0.4385449	-5.5129496	-0.5678526	-0.6101846	-0.643887	-0.6713876

The above table shows that both Skewness and kurtosis decrease with increasing σ, since the PDF of RU{LL} is Skewed to the left as can be shown from the graph of the PDF.

4.2 The Generating Function χ_u.

The generating function χ_u is the solution of the equation

$$\chi_u = G(u)$$

Where u is the uniform random variable u(0, 1), or u is a vector of uniform random variables u (0, 1)

Then χ_u is given by:

$$\chi_u = \frac{a + b\sqrt{2}\,\sigma\left[\ln\left(\frac{1}{1-u}\right)\right]^{1/2}}{1 + \sqrt{2}\,\sigma\left[\ln\left(\frac{1}{1-u}\right)\right]^{1/2}} \qquad (4.2)$$

Where u is a vector of uniformly in dependent u(0,1) random variables.

5- Expanding g(x) in terms of f(x), F(x) of the Uniform Distribution and r(x) of Rayleigh Distribution.

From (3.3)

$$g(x) = f(x)r\left\{\frac{F(x)}{1-F(x)}\right\}(1-F(x))^{-2}$$

$$= f(x)r\left\{\frac{x-a}{b-x}\right\}(1-F(x))^{-2}$$

Since $|F(x)| < 1$, then:

$$g(x) = f(x)r \left\{\frac{x-a}{b-x}\right\} \sum_{j=0}^{\infty} (j+1)F^j(x)$$

$$= \frac{1}{\sigma^2\,(b-a)} \left(\frac{x-a}{b-x}\right) e^{-(\frac{x-a}{b-x})/2\sigma^2} \sum_{j=0}^{\infty} (j+1)F^j(x)$$

Where: $f(x) = \frac{1}{b-a}$ and $F(x) = \frac{x-a}{b-a}$

6- The Ordinary Moments of RU{LL} Distribution

The rth ordinary moment of the RU{LL} distribution is given by:

$$\mu_r' = E\,(x^r) = \int_a^b x^r\; g(x)dx \tag{6.1}$$

Where g(x) as given by (3.9).

Integrating (6.1) using the integrate (.) function of the R software we can obtain all the moments of the RU{LL} random variable χ. Table (2) given below presents the values of the first four moments, the variance, the coefficient of variation (CV), the skewness (SK) and the kurtosis (kur) of RU{LL} distribution for various values of a, b, and σ.

Table(2): The values of the mean $(\mu), \mu_2', \mu_3', \mu_4', variance\ (\mu_2)$, skewness (sk) and kurtosis (kur) of RU{LL} for various values of a, b and σ.

Table (2): the values of the moment for a=0, b=3 and σ= 1, 2, 3, 4

σ / Moments	$\sigma = 1$	$\sigma = 2$	$\sigma = 3$	$\sigma = 4$
Mean (μ)	0.1610105	0.02199681	0.004824066	0.001457297
$m2\ (\mu_2')$	0.1818124	0.02243712	0.004194036	0.001071796
$m3\ (\mu_3')$	0.2181759	0.02481385	0.004036417	0.0008824280
$m4\ (\mu_4')$	0.2722950	0.02895977	0.004170466	0.0007876759
$variance\ (\mu_2)$	0.1558880	0.02195326	0.004170764	0.001059672
CV	2.4521790	6.73581000	0.1338735	0.2244281
Sk	2.2535470	7.17996100	0.1476104	0.2508961
Kur	3.3536050	52.69280000	0.02323031	0.06809246

7- The Mean Residual Life (MRL)

In reliability studies, the mean residual life or life expectancy is an important characteristic of the study. The expected additional lifetime given that a component has survived until time t is a function of t, called the mean residual life. If the random variable χ represents the life of a component, then the MRL is given by $m(t) = E(x - t|x > t)$.

The MRL has been studied in life length studies by Bryson and Siddiqui [5], Muth [23] and Bradley, et. Al. [4].

If X is a nonnegative random variable representing the life of a component having the RU{LL} distribution, the MRL is defined by:

$$m(t) = E\ (x - t|x > t)$$

$$= \frac{1}{P_r\ (x > t)} \int_t^\infty (x - t)\ \ g(x)dx, t\ \geq 0$$

Where g(.) is the PDF of RU{LL} distribution. Writing $X - t = \int_t^x du$ and employing Tonelli's therom yields the following formula:

$$m(t) = \frac{1}{S(t)} \int_t^\infty S(t)dt, s(t) > 0 \tag{7.1}$$

The MRL of equation (7.1) is called the theoretical mean residual life denoted by TMRL.

The empirical MRL, named EMRL can be computed from a random sample of size n drawn from a RU{LL} distribution as follows: TMRL = expected remaining life after the unit survived until life t

$$= \sum_{k=1}^{n} \left[x_{(k)}\colon xn - x_{(k)}\right] / (n - k)$$

Where

$x_{(k)}\ \leq t\ <\ x_{(k+1)}\ , x_{(k)} = the\ kth$ order statistic.

The following table provides the first 10 rows of the empirical MRL (EMRL) and the theoretical MRL (TMRL) of a random sample of size n = 50 drawn from RU{LL} population with parameters a=0 , b=3 and $\sigma = 2$

Table(3): EMRL and TMRL for different values of death time (a=0, b=3, $\boldsymbol{\sigma = 2}$)

No	Death time	EMRL	TMRL
1	0.9260461	1.1403988	0.36849468
2	1.1088163	0.9775792	0.27763928
3	1.1375127	0.9690718	0.26480715
4	1.4473341	0.6735819	0.14972570
5	1.5669124	0.5663148	0.11605129

6	1.5783727	0.5674648	0.11311216
7	1.5904157	0.5683386	0.11007586
8	1.6048469	0.5670957	0.10650713
9	1.7483093	0.4339658	0.07484403
10	1.7764270	0.4159943	0.06932960

Both EMRL and TMRL decreases with increasing death time

8- Renyi entropy

The entropy of a random variable is a measure of uncertainty variation and is used to measure the randomness of systems. It has been used in various situations in science and engineering, [17,27,28,29].

If x has RU{LL} (a, b, σ), then the Renyi entropy is defined by:

$$I_R(p) = \frac{1}{(1-p)} \, ln \left(\int_a^b g_{(x)}^{p} \; dx \right)$$

$$P \; > 0 \, , P \neq 1$$

The following table gives the Renyi entropy ($I_R(p)$) values for a=0, b=3, P=2 and 7 different values of σ.

Table (4): $\boldsymbol{I_R(p)}$ values for a=0, b=3 $\boldsymbol{\sigma = 2}$ and P=2

$\boldsymbol{I_R(p)}$	7.615527	10.20296	12.08857	13.56803
	14.7865	15.82279	16.72456	

9- Methods of Estimation

In this section we estimate the parameters of the RU{LL} distribution by 4 different methods using complete sample technique. These methods are: maximum likelihood (MLE), Least-squares (LS), weighted least squares and percentile based estimation. The performance of all methods are studied by the R software.

9.1 Maximum likelihood estimation (MLE):

The MLE method is a general method and its estimators have some optimum properties such as consistency, asymptotic efficiency and invariance property.

Let $\chi_1, \chi_2, \dots, \chi_n$ be a random sample from RU{LL} population with PDF g(x) given in (3.9) with unknown parameters a, b and σ and the log-likelihood function is l(a, b, σ) then the MLE estimates of a, b and σ are the simultaneously solution of the following equations:

$$\frac{\partial \; l(a,b,\sigma)}{\partial a} = 0 \, , \frac{\partial \; l(a,b,\sigma)}{\partial b} = 0 \, , \frac{\partial \; l(a,b,\sigma)}{\partial J} = 0$$

Which gives the MLE estimates $(\hat{a}\,, \hat{b}, \hat{\sigma})$. These equations are solved numerically using R software.

9.2 Method of Ordinary Least Squares (LS)

The best estimates according to LS method are those which minimize the following quantity:

$$Q_1 = \sum_{i=1}^{n} \left(G\ \left(x_{(i)} - \frac{i}{n+1} \right) \right)^2$$

With respect to a, b and σ.

Where $x_{(i)}$ is the ith orders statistic of RU{LL}

9.3 Method of Weighted Least Squares (WLS):

The WLS estimators of a, b and σ of RU{LL} distribution can be obtained by minimizing the quantity:

$$Q_2 = \sum_{i=1}^{n} \frac{(n+1)^2\ (n+2)}{(n-i+1)} \left(G\ \left(x_{(i)} - \frac{i}{n+1} \right) \right)^2$$

With respect to a, b and σ.

9.4 Method of Percentile Estimation (PCE)

This method introduced by kao [13,14] the PCEs estimators of a, b and σ of RU{LL} distribution can be obtained by minimizing the quantity:

$$Q = \sum_{i=1}^{n} \left[x_{(i)} - a\ \left(\frac{1-(1-\sigma)(i/(n+1))}{1-\ (i/(n+1)} \right)^{1/\sigma} \right]^2$$

Where $x_{(i)}$ is the ith order statistic of RU{LL} distribution.

9.5 Simulation Results

We introduce Monte Carlo simulation results, using R software. The above 4 methods are applied to estimate the parameters a, b and σ of RU{LL} distribution. The number of Monte Carlo replications was N=5000 for samples of size n = (30, 100, 200) for a = (0, 1), b= (2, 3) and σ = (0.5, 2). The average values of estimates (AVEs) and mean square error (MSEs) of MLEs, LSEs, WLSEs and PCEs are obtained and displayed in table (5-10). From these tables we observe that the MSEs decreases as sample size increase, which shows the consistency of all estimates. The results show that PCE gives the best results, for estimating a, b and σ, in terms of MSEs in all the cases.

The ordering of performance of estimates in terms of MSEs (from best to worst) is PCEs, WLSEs, LSEs, MLEs.

Tables of the estimates of the parameters using the four methods of estimation according to certain values of the parameters.

Table (5): The AVEs and their corresponding MSEs (in parentheses)

For n = 30 and a = 0

Parameters	MLEs	LSEs	WLSEs	PCEs
$b = 2$ $\sigma = 0.5$	2.8999 (0.0100) 0.000 (9.0000)	3.0578 (0.0346) 0.3688 (6.9375)	3.0364 (0.0223) 0.3573 (6.2033)	3.1365 (0.0825) 2.7617 (1.2501)
$b = 2$ $\sigma = 2$	1.8001 (0.0487) 0.000 (4.0000)	2.0415 (0.0288) 0.5451 (2.1508)	2.0255 (0.0198) 0.5304 (2.1862)	2.0879 (0.0462) 1.9124 (0.5081)
$b = 3$ $\sigma = 0.5$	1.8970 (1.2375) 0.000 (0.2500)	3.6780 (7.2572) 2.4530 (5.9205)	3.4367 (4.2816) 2.3505 (5.1271)	3.5646 (3.7957) 0.4853 (0.0526)
$b = 3$ $\sigma = 2$	4.9443 (3.7805) 0.000 (4.0000)	3.0656 (0.0663) 0.5485 (2.1409)	3.0409 (0.0452) 0.5332 (2.1778)	3.1276 (0.1019) 1.9248 (0.5484)

Table (6): The AVEs and their corresponding MSEs (in parentheses)

For n = 30 and a = 1

Parameters	MLEs	LSEs	WLSEs	PCEs
$b = 2$ $\sigma = 0.5$	1.8618 (0.0239) 0.000 (9.2500)	2.2693 (1.5329) 2.4806 (6.1137)	2.1948 (1.4870) 2.4098 (5.6316)	2.2165 (0.6649) 0.4813 (0.0541)
$b = 2$ $\sigma = 2$	2.0326 (0.0179) 0.000 (4.0000)	2.0220 (0.0073) 0.5478 (2.1430)	2.0135 (0.0050) 0.5323 (2.1808)	2.0440 (0.0117) 1.9187 (0.5374)
$b = 3$ $\sigma = 0.5$	2.2787 (0.5351) 0.000 (0.2500)	3.5368 (5.9008) 2.5073 (6.2264)	3.3136 (2.1930) 2.3944 (5.3702)	3.3912 (1.8589) 0.4775 (0.0502)
$b = 3$ $\sigma = 2$	4.9231 (3.7450) 0.000 (4.0000)	3.0443 (0.0275) 0.5470 (2.1429)	3.0275 (0.0189) 0.5332 (2.1811)	3.0882 (0.0464) 1.9154 (0.5098)

Table (7): The AVEs and their corresponding MSEs (in parameters)

For n = 100 and a = 0

Parameters	MLEs	LSEs	WLSEs	PCEs
$b = 2$ $\sigma = 0.5$	2.043 (0.0217) 0.000 (0.250)	2.094 (0.1526) 2.170 (3.273)	2.0510 (0.0904) 2.0984 (2.8806)	2.0939 (0.0934) 0.4896 (0.0145)
$b = 2$ $\sigma = 2$	1.8014 (0.0447) 0.000 (4.000)	0.0120 (0.0058) 0.5127 (2.2193)	2.0054 (0.0041) 0.5066 (2.2360)	2.0298 (0.0098) 1.9567 (0.151)
$b = 3$ $\sigma = 0.5$	2.0217 (0.9668) 0.000 (0.2500)	3.1446 (0.3716) 2.1735 (3.3205)	3.0770 (0.2086) 2.0993 (2.890)	3.1405 (0.2134) 0.4896 (0.0144)
$b = 3$ $\sigma = 2$	4.9443 (3.7805) 0.000 (4.000)	3.0187 (0.0137) 0.5129 (2.2186)	3.0072 (0.0097) 0.5059 (2.2382)	3.0431 (0.0224) 1.9613 (0.1522)

Table (8): The AVEs and their corresponding MSEs (in parentheses)

For n = 100 and a = 1

Parameters	MLEs	LSEs	WLSEs	PCEs
$b = 2$ $\sigma = 0.5$	1.8402 (0.0260) 0.000 (0.2500)	2.0428 (0.0344) 2.1572 (3.2019)	2.0227 (0.0206) 2.0892 (2.8275)	2.0440 (0.2142) 0.4906 (0.0141)
$b = 2$ $\sigma = 2$	2.0630 (0.0070) 0.000 (4.000)	2.0058 (0.0015) 0.5127 (2.2194)	2.0023 (0.0010) 0.5063 (2.2370)	2.0147 (0.0024) 1.9576 (0.1522)
$b = 3$ $\sigma = 0.5$	2.3893 (0.3849) 0.000 (0.2500)	3.0885 (0.1407) 2.1591 (3.2107)	3.0448 (0.0826) 2.0854 (2.8124)	3.0871 (0.0861) 0.4917 (0.0144)
$b = 3$ $\sigma = 2$	4.9435 (3.7790) 0.000 (4.000)	3.0125 (0.0062) 0.5140 (2.2158)	3.0056 (0.0044) 0.5076 (2.231)	3.0303 (0.0101) 1.9535 (0.1540)

Table (9): The AVEs and their corresponding MSEs (in parentheses)

For n = 200 and a = 0

Parameters	MLEs	LSEs	WLSEs	PCEs
$b = 2$ $\sigma = 0.5$	2.0434 (0.0018) 0.000 (0.2500)	2.0413 (0.0509) 2.0765 (2.6543)	2.0209 (0.0339) 2.0423 (2.5027)	2.0473 (0.0349) 0.4924 (0.0066)
$b = 2$ $\sigma = 2$	1.7958 (0.0450) 0.0000 (4.0000)	2.0058 (0.0028) 0.5064 (2.2343)	2.0020 (0.0020) 0.5029 (2.2441)	2.0160 (0.0045) 1.9735 (0.0774)
$b = 3$ $\sigma = 0.5$	2.0698 (0.8727) 0.0000 (0.2500)	3.0706 (0.1230) 2.0867 (2.6990)	3.0360 (0.0791) 2.0480 (2.5259)	3.0740 (0.0809) 0.4919 (0.0069)
$b = 3$ $\sigma = 2$	4.9443 (3.7805) 0.000 (4.0000)	3.0081 (0.0067) 0.5058 (2.2362)	3.0018 (0.0045) 0.5019 (2.2470)	3.0213 (0.0100) 1.9807 (0.0770)

Table (10): The AVEs and their corresponding MSEs (in parentheses)

For n = 200 and a = 1

Parameters	MLEs	LSEs	WLSEs	PCEs
$b = 2$ $\sigma = 0.5$	1.8380 (0.0262) 0.0000 (9.2500)	2.2198 (1.0134) 2.0711 (2.6475)	2.0085 (0.0087) 2.0322 (2.4799)	2.0216 (0.0089) 0.4952 (0.0070)
$b = 2$ $\sigma = 2$	2.0784 (0.0080) 0.0000 (4.0000)	2.0026 (0.0007) 0.5055 (2.2371)	2.0006 (0.0005) 0.5019 (2.2473)	2.0066 (0.0011) 1.9851 (0.0781)
$b = 3$ $\sigma = 0.5$	2.4429 (0.3156) 0.0000 (0.2500)	3.0444 (0.0554) 2.0805 (2.6817)	3.0218 (0.0361) 2.0425 (2.5110)	3.4813 (0.0365) 0.4930 (0.0070)
$b = 3$ $\sigma = 2$	4.9443 (3.7805) 0.0000 (4.0000)	3.0052 (0.0027) 0.5058 (2.2358)	3.0013 (0.0019) 0.5022 (2.2460)	3.0145 (0.0044) 1.9779 (0.0741)

Conclusion

We introduced a new distribution, namely, RU {LL} distribution. The main structural properties of the new distribution are studied.

The parameters of the distribution are estimated using four methods of estimation, and the performance of these methods is evaluated using the R software.

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