Kumaraswamy Muth Distribution: Properties and Applications

Mahmoud Ali Selim
Faculty of Commerce, Al-Azhar University, Egypt.

Tamer Hassan Mohammed Ali
Faculty of Commerce, Al-Azhar University, Egypt.

Mohamed Abdelkader
Faculty of Commerce, Al-Azhar University, Egypt.

Abstract

We presented a new three-parameter distribution known as the Kumaraswamy Muth distribution (KMD) in this study. It extends the Muth distribution, a continuous random variable widely used in reliability theory. The KMD distribution and its mathematical properties are examined, including the derivation of the quantile function, probability density and distribution functions, moments, generating function, and order statistics. Model parameters are estimated using the maximum likelihood method, and practical applications are utilized to illustrate the versatility of the KMD distribution.

Keywords: Muth distribution, maximum likelihood estimation; moment generating function; hazard rate; Kumaraswamy distribution function; order statistics and quantile function

Introduction

Muth [1] introduced a continuous probability distribution suitable for reliability analysis. When the probability density function is defined as such, a random variable Y is characterized as having a Muth distribution with a parameter.

$$f(y;\alpha) = (e^{\alpha y} - \alpha)\, exp\left(\alpha y - \frac{1}{\alpha}(e^{\alpha y} - 1)\right), \quad y > 0 \qquad (1)$$

The cumulative distribution function of Y , is the following :

$$F(y;\alpha) = 1 - \, exp\left(\alpha y - \frac{1}{\alpha}(e^{\alpha y} - 1)\right), \qquad y > 0 \qquad (2)$$

Where $\alpha \in (0\,,1)$ is a shape parameter. The following facts help to explain the Muth distribution: [2]

- As λ tends toward 0, it resembles the standard exponential distribution with a parameter of 1.
- In the right tail, it exhibits less probability mass than common gamma, log-normal, and Weibull distributions.
- This distribution conforms to the requirements for generating random variables. It also upholds the relationship between mode, mean, and median.
- Additionally, it offers sample flexibility to handle unique lifetime datasets, particularly those derived from reliability experiments.

Kumaraswamy (1980) [3] introduced a two-parameter probability distribution over the interval (0, 1) as an alternative to the beta distribution. This distribution is known as the Kumaraswamy distribution and is denoted as Kum (a, b). The probability density function for this distribution can be defined as follows:

$$f_{Kum}(y) = \; aby^{a-1}(1 - y^{a})^{b-1} \quad 0 < y < 1 \quad (a,b) > 0 \quad (3)$$

, and cumulative function is

$$F_{kum} = 1 - (1 - y^{a})^{b} \quad 0 < y < 1, \quad (a,b) > 0 \qquad (4)$$

Jones (2009) [4] demonstrated that, much like the beta distribution, the Kumaraswamy distribution possesses characteristics such as being unimodal, uniantimodal, and exhibiting properties of both increasing and decreasing density depending on the parameter values. Additionally, he pointed out that the Kumaraswamy distribution offers advantages for simulation studies due to its straightforward cumulative distribution function and quantile function, as well as a simple normalizing constant and explicit formulas for the distribution and quantile function that do not rely on special functions. It also provides a simple formula for generating random variables and computing L-moments.

In contrast, Jones highlighted some advantages of the beta distribution, such as being a one-parameter sub-family of symmetric distributions, simpler moment estimation, and more versatile methods for generating the distribution through physical processes. The beta distribution also offers a simpler formula for moments and moment generating functions.

Due to these considerations, Cordeiro and de Castro (2011) [5] introduced the beta-generated family as an alternative to the beta distribution, employing the Kumaraswamy distribution instead. This family is defined for an arbitrary baseline cumulative distribution function and probability density function (pdf) in a generalized form.

$$F(y; a, b) = 1 - (1 - G(y)^a)^b \quad (5)$$

$$f(y; a, b) = ab\, g(y) G(y)^{a-1} (1 - G(y)^a)^{b-1} \quad (6)$$

In the forthcoming sections, we will explore a novel generalized distribution named the Kumaraswamy Muth distribution (KMD). This distribution is formed by introducing Kumaraswamy (1980), Cordeiro and de Castro (2011), and Muth distributions. We will examine the statistical properties of KMD.

The structure of this paper is as follows: In the next section, we will introduce the cumulative distribution function (cdf), probability density function (pdf), and hazard rate functions of the KM distribution, accompanied by corresponding figures.

Section 3 will be further divided into five parts:

Section 3.1 will delve into the expansion of the cdf and pdf of the KM distribution.

In Section 3.2, we will present the quantile function of the KM distribution.

Section 3.3 will focus on the presentation of r-th moments of the KM distribution.

The final part of Section 3 will discuss order statistics functions.

In Section 4, we will employ the maximum likelihood method to estimate the model parameters.

Section 5 will demonstrate the practical application of the KM distribution using a real dataset.

Lastly, the conclusion will be presented in the last section.

2. The Kumaraswamy Muth distribution:

In this section the cdf ,pdf and hazard functions of Kumaraswamy Muth distribution(KMD) will be introduced by setting the Muth bassline function (1.1) and (1.2) in equations (1.5) an(1.6), then the cdf and pdf of the KMD are obtained as follow:

$$F_{KM}(y) = 1 - \left\{1 - \left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a}\right\}^{b}, \quad a, b, \propto > 0. \quad y > 0 \qquad (7)$$

and

$$f_{KM}(y) = ab(e^{\propto y} - \propto)\exp\left(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right)\left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a-1}$$

$$\times \left\{1 - \left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a}\right\}^{b-1}, \quad a, b, \propto > 0. \quad y > 0 \ , \qquad (8)$$

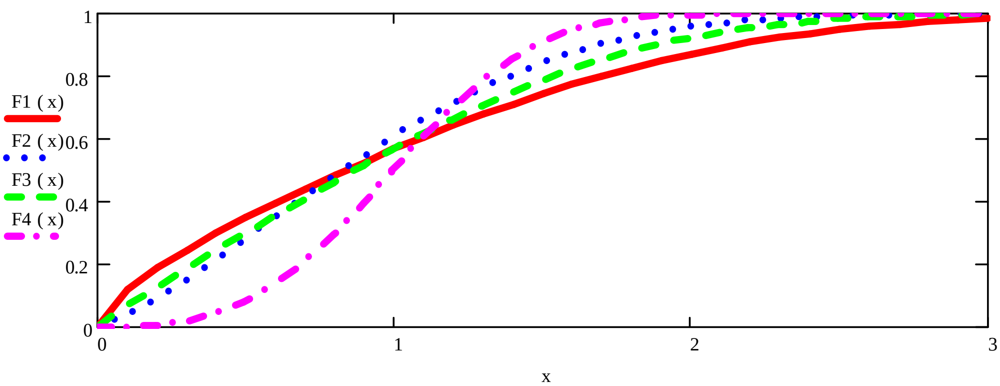

Figure 1 Cumulative Distribution Function of Kumaraswamy Muth Distribution

The concept of failure rates, also known as hazard rates, and survival functions plays a crucial role in various domains, including industry, engineered systems, finance, and forms the foundation for designing social security, medical insurance, and safety systems across diverse applications. The survival and hazard rate functions for a random variable Y following the Kumaraswamy distribution can be expressed as follows:

$$S(y) = 1 - F(y) = \left\{1 - \left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a}\right\}^{b}$$

$$h(y) = \frac{f(y)}{S(y)}$$

$$h(y) = \frac{ab(e^{\propto y} - \propto)\exp\left(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right)\left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a-1}}{1 - \left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a}}, \quad y > 0 \quad (9)$$

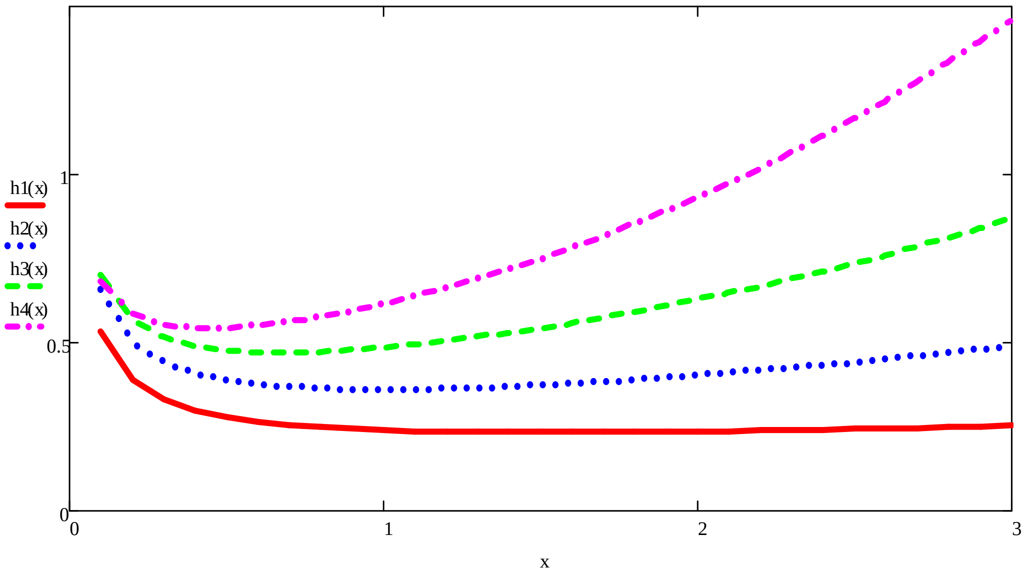

Figure 2 Hazard Rate of Kumaraswamy Muth

3. The statistical Properties:

The quantile function, random variable generation function, moments, moment generating function, skewness, kurtosis, and order statistics are some of the statistical properties of the KM distribution that are obtained in this section.

3.1 Expansions for the cumulative and density functions [5]

The expansion for the cumulative distribution function of KM distribution can be derived by using the generalized binomial theorem. For any real number r > 0 and |z| < 1 the binomial expansion is

$$(1-z)^r = \sum_{i=0}^{\infty}(-1)^i \binom{r}{i} z^i$$

where $\binom{r}{i} = \frac{r(r-1)\ldots\ldots(r-i+1)}{i!}$

Using the binomial expansion in equation (2.1), we get the cdf as a power series expansion as follows

$$F_{KM}(y) = 1 - \sum_{i=0}^{\infty}(-1)^i \binom{b}{i} \left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{ia}$$

using the binomial expansion, again in the last term of above equation , we get the expansion of cdf as follow

$$F_{KM}(y) = 1 - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j} \binom{b}{i}\binom{ai}{j} e^{j\left[\alpha y - \frac{1}{\alpha}(e^{\propto y}-1)\right]} \quad (10)$$

Differentiating (11) with respect to x gives the expansion of pdf as follow

$$f_{KM} = (e^{\propto y} - \propto)\sum_{i=0}^{\infty}\sum_{0}^{\infty}(-1)^{i+j} \binom{b}{i}\binom{ai}{j} j e^{j\left[\alpha y - \frac{1}{\alpha}(e^{\propto y}-1)\right]} \quad (11)$$

Figure 1 shows the different shapes of KM distribution pdfs, the shapes can be positively skewed. Figure 2 shows the cdf of KM distribution.

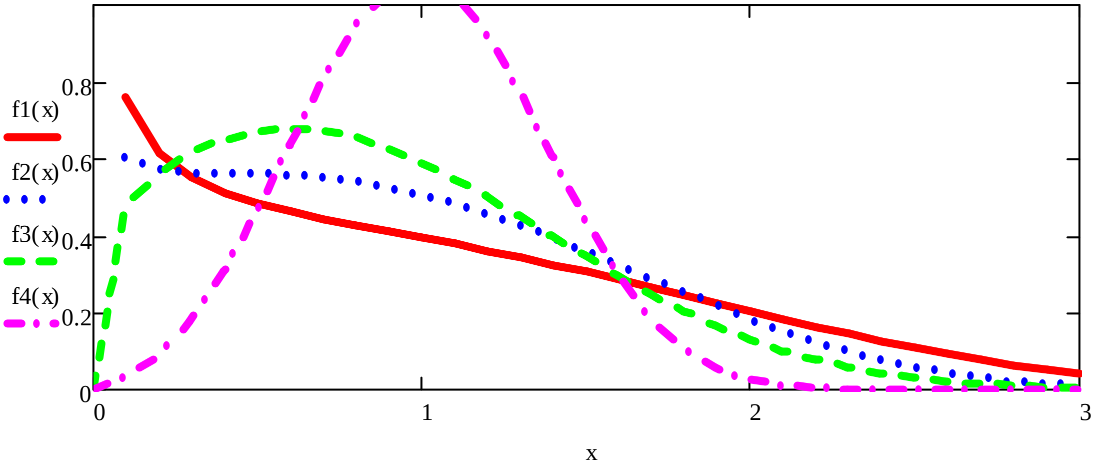

Figure 3 Probability Density Function of Kumaraswamy Muth Distribution

3.2 Quantile function:[6]

The quantile function of KMD distribution can be obtained by inverting (7) as follow

$$F_{KM}(y) = 1 - \left\{1 - \left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a}\right\}^{b}, \quad a, b, \propto > 0. \quad y > 0$$

$$q(u) = F^{-1}{}_{KM}(y)$$

$$q = 1 - \left\{1 - \left[1 - exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a}\right\}^{b}$$

$$(1-q)^{\frac{1}{b}} = 1 - \left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a}$$

$$\left[1 - (1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}} = 1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)$$

$$1-\left[1-(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}} = exp(\propto y-\frac{1}{\propto}(e^{\propto y}-1)$$

$$ln\left\{1-\left\{1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}\right\} =\propto y-\frac{1}{\propto}(e^{\propto y}-1)$$

$$\propto y-\frac{1}{\propto}e^{\propto y} = ln\left\{1-\left\{1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}\right\}-\frac{1}{\propto} \qquad (12)$$

By using the LEMMA 1 (P. Jodr´a.)[7]

$$y=\frac{1}{\propto}\left[\ln\left\{1-\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)\right\}-\frac{1}{\propto}\right]-\frac{1}{\propto}W\left(-\frac{1}{\propto}exp\left[\ln\left\{1-\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)\right\}-\frac{1}{\propto}\right]\right)$$

$$y=\frac{1}{\propto}\left[\ln\left\{1-\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)\right\}-\frac{1}{\propto}\right]-\frac{1}{\propto}W\left(-\frac{1-\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)}{\propto e^{\frac{1}{\propto}}}\right)$$

$$y=\frac{1}{\propto}\left[\ln\left\{1-\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)\right\}-\frac{1}{\propto}\right]-\frac{1}{\propto}W\left(\frac{\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)-1}{\propto e^{\frac{1}{\propto}}}\right) \qquad (13)$$

Additionally, for any $\alpha \in (0,1]$, $q \in (0,1]$ and $x > 0$, it can be checked that $\frac{\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)-1}{\propto e^{\frac{1}{\propto}}} \in (\frac{-1}{e},0)$, since $\lim_{\propto\to 0^+}\frac{\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)-1}{\propto e^{\frac{1}{\propto}}} = 0$, and also that $\ln\left\{1-\left\{1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}\right\}-\frac{1}{\propto}-\propto y < -1$, which infer that the Lambert W function in Eq. (above) correlate to the negative branch W_{-1}.[8]

Then, let Y be a random variable following Kumaraswamy Muth distribution with parameters a, b> 0 and

$\alpha \in (0,1]$. The quantile function of Y, $Q(u;\propto,a,b)$, is

$$Q(u;\propto,a,b) = \frac{1}{\propto}\left[\ln\left\{1-\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)\right\}-\frac{1}{\propto}\right]-\frac{1}{\propto}W_{-1}\left(\frac{\left(1-\left[(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)-1}{\propto e^{\frac{1}{\propto}}}\right) \quad (14)$$

The the median M(y) of KM distribution can be calculated by putting q=0.5 as follows

$$M(Y) = \frac{1}{\propto}\left[\ln\left\{1-\left(1-\left[(1-0.5)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)\right\}-\frac{1}{\propto}\right]-\frac{1}{\propto}W_{-1}\left(\frac{\left(1-\left[(1-0.5)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)-1}{\propto e^{\frac{1}{\propto}}}\right) \quad (15)$$

By using Wolfram Alpha (https://www.wolframalpha.com/) we can obtain result of the median. Also by putting q=0.25 and q= 0.75 we can be obtained the quartiles.

3.3 Skewness and kurtosis:

The statistical measures of skewness and kurtosis consider important role in describing the shape parameters of the probability distributions. Based on quartiles Bowley's introduced skewness measure (Kenney and keeping, (1962) as follows

$$Sk = \frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \quad (16)$$

and the Moors' kurtosis measure based on octiles (Moors (1988)) is given by

$$Ku = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})} \qquad (17)$$

3.3 generating random variables:

When ; $\propto, a \text{ and } b$ are known the quantile function of KM distribution can be used to generate KM random variables as follows

$$y = \frac{1}{\propto}\left[\ln\left\{1 - \left(1 - \left[(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)\right\} - \frac{1}{\propto}\right] - \frac{1}{\propto}W_{-1}\left(\frac{\left(1 - \left[(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right) - 1}{\propto e^{\frac{1}{\propto}}}\right) \qquad (18)$$

Where, u is generated number from the Uniform distribution (0, 1)

3.4 Moments

If Y has the KM distribution, then the The r-th moment of KM distribution will be given by following steps:

$$\boldsymbol{\mu_r^{/} = E(y^r) = \int_0^\infty y^r f(y) dy}$$

$$\mathbf{E(y^r)} = \int_0^\infty \sum_{i=0}^{\infty}\sum_{0}^{\infty}(-1)^{i+j}\binom{b}{i}\binom{ai}{j} j\mathbf{y^r}(e^{\propto y} - \propto)e^{j\left[\alpha y - \frac{1}{\alpha}(e^{\propto y}-1)\right]}\,\mathbf{dy}$$

$$\mathbf{E(y^r)} = \sum_{i=0}^{\infty}\sum_{0}^{\infty}(-1)^{i+j}\binom{b}{i}\binom{ai}{j} j\int_0^\infty \mathbf{y^r}(e^{\propto y} - \propto)e^{j\left[\alpha y - \frac{1}{\alpha}(e^{\propto y}-1)\right]}\,\mathbf{dy} \qquad \mathbf{(19)}$$

Put $\qquad \boldsymbol{\omega_{ij}} = \sum_{i=0}^{\infty}\sum_{0}^{\infty}(-1)^{i+j}\binom{b}{i}\binom{ai}{j} j$

$\boldsymbol{\frac{e^{\alpha y}}{\alpha} = x}$ **then** $\boldsymbol{y = \frac{ln(\alpha x)}{\alpha}}$, $\boldsymbol{\frac{1}{\alpha} < x < \infty}$ **and** $\boldsymbol{dy = \frac{dx}{\alpha x}}$

By substituting in Eq(19) :

$$E(y^r) = \omega_{ij}\alpha^{j-r}e^{\frac{j}{\alpha}}\int_{\frac{1}{\alpha}}^{\infty}(x-1)x^{j-1}e^{-xj}(\ln(\alpha x))^r dx$$

Ozlap and Bairamove [9] provide the following integrals, and it is noteworthy that these integrals bear a relation to the exponential integrals distribution, as observed in the work of Meijer and Baken.3.5 Order statistics

Let $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ are the order statistics of a random sample follows a continuous distribution with cdf $F(x)$ and pdf $f(x)$. Then the pdf of $X_{(k:\mathrm{n})}$ is

$$f_{k:n}(x) = \frac{n!}{(k-1)!\,(n-k)!} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}, \quad k = 1,2,\dots,n$$

Then, the pdf of the k-th order statistics of the KM distribution is

$$f_{k:n}(y) = \frac{n!\,ab}{(k-1)!\,(n-k)!}(e^{\propto y} - \propto) \exp\left(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right)\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a-1}$$
$$\times \left\{1-\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a}\right\}^{b-1}\left\{1-\left\{1-\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a}\right\}^{b}\right\}^{k-1}$$
$$\times \left\{\left\{1-\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a}\right\}^{b}\right\}^{n-k} \qquad (20)$$

if $k = 1$, the pdf of order statistics is

$$f_{1:n}(y) = nab(e^{\propto y} \propto) \exp\left(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right)\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a-1}\left\{1-\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a}\right\}^{b-1}$$
$$\times \left\{\left\{1-\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a}\right\}^{b}\right\}^{n-1} \qquad (21)$$

and if $k = \mathrm{n}$, the pdf of order statistics is

$$f_{n:n}(y) = nab(e^{\propto y} - \propto) \exp\left(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right)\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a-1}$$
$$\times \left\{1-\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a}\right\}^{b-1}$$
$$\times \left\{1-\left\{1-\left[1-\exp(\propto y - \frac{1}{\propto}(e^{\propto y}-1)\right]^{a}\right\}^{b}\right\}^{n-1} \qquad (22)$$

4. Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) and approximate confidence intervals for the unknown parameters of the KM distribution will be discussed in this section. Let $Y_1, Y_1, \dots, Y_n$ is a random sample of size n from the KM distribution with density function:

$$f_{KM}(y) = ab(e^{\propto y} - \propto)\exp\left(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right)\left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a-1}$$

$$\times \left\{1 - \left[1 - \exp(\propto y - \frac{1}{\propto}(e^{\propto y} - 1)\right]^{a}\right\}^{b-1} \quad , \quad a, b, \propto > 0. \quad y > 0$$

Then the likelihood function (LF) is

$$\mathcal{L} = a^n b^n \prod_{i=1}^{n} \left\{ (e^{\propto y_i} - \propto)\exp\left(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right)\left[1 - \exp\left(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]^{a-1} \times \left\{1 - \left[1 - \exp\left(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]^{a}\right\}^{b-1}\right\} \tag{23}$$

and the log-likelihood function ($ln\mathcal{L}$) is

$$ln\mathcal{L} = nln(ab) + (a-1)\sum_{i=1}^{n} ln\left[1 - \exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right] + (b-1)\sum_{i=1}^{n} ln\left[1 - \left[1 - \exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]^{a}\right] + \sum_{i=1}^{n} ln(e^{\propto y_i} - \propto) + \sum_{i=1}^{n}\left(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right) \tag{24}$$

Then, the maximum likelihood estimators (MLE) of $\hat{b}$ is

$$\hat{b} = -\frac{-n}{\sum_{i=1}^{n} ln\left[1 - \left[1 - \exp(\hat{\propto} y_i - \frac{1}{\hat{\propto}}(e^{\hat{\propto} y_i} - 1)\right]^{\hat{a}}\right]} \tag{25}$$

where $\hat{\propto}$ and $\hat{a}$ are the MLEs of the parameters α and a, This can be derived by solving the non-linear equations below.

$$\frac{\partial ln\mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} \frac{(a-1)\left\{-\left[y_i - \left(\frac{y_i e^{\propto y_i}}{\propto} - \frac{e^{\propto y_i}}{\propto^2}\right)\right] - \frac{1}{\propto^2}\right\}\left[\exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]}{1 - \exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)}$$
$$+ \sum_{i=1}^{n} \frac{(b-1)\left\{-a\left[1 - \exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]^{a-1}\right\}\left\{-\left[y_i - \left(\frac{y_i e^{\propto y_i}}{\propto} - \frac{e^{\propto y_i}}{\propto^2}\right)\right] - \frac{1}{\propto^2}\right\}\left[\exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]}{1 - \left[1 - \exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]^{a}}$$
$$+ \sum_{i=1}^{n} \frac{y_i e^{\propto y_i} - 1}{e^{\propto y_i} - \propto} + \sum_{i=1}^{n} \left[y_i - \left(\frac{y_i e^{\propto y_i}}{\propto} - \frac{e^{\propto y_i}}{\propto^2}\right)\right] - \frac{1}{\propto^2} \qquad (26)$$

$$\frac{\partial ln\mathcal{L}}{\partial \mathrm{a}} = \frac{\mathrm{n}}{\mathrm{a}} + \sum_{i=1}^{n} ln\left[1 - \exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]$$
$$- \sum_{i=1}^{n} \frac{\left[1 - \exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]^{a} ln\left(1 - \exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right)}{1 - \left[1 - \exp(\propto y_i - \frac{1}{\propto}(e^{\propto y_i} - 1)\right]^{a}} \qquad (27)$$

By using iterative techniques like the Newton-Raphson algorithm the equations $\hat{b}$, $\frac{\partial ln\mathcal{L}}{\partial \alpha}$ and $\frac{\partial ln\mathcal{L}}{\partial \mathrm{a}}$ can be solved numerically

5. Application[10]

In this section the real data will be introduced as an application of KM distribution . the real data set presented by Bader and Priest (1982), the data represent the strength data measured in GPA, for single carbon fiber were tested under pressure at gauge time lengths of 20 mm with sample size =63. The data are introduced in the following table:

Table 1. Data set (gauge time lengths of 20 mm.)

1.312	1.314	1.479	1.552	1.7	1.803	1.861
1.865	1.944	1.958	1.966	1.997	2.006	2.021
2.027	2.055	2.063	2.098	2.140	2.179	2.224
2.24	2.253	2.27	2.272	2.274	2.301	2.301
2.359	2.382	2.382	2.426	2.434	2.435	2.478
2.49	2.511	2.514	2.535	2.554	2.566	2.57
2.586	2.629	2.633	2.642	2.648	2.684	2.697
2.726	2.77	2.773	2.8	2.809	2.818	2.821
2.848	2.880	2.809	2.818	2.821	2.848	2.88

The KM distribution will be fitted with above data, inverse power Muth (IPM)[11] and truncated Kumaraswamy exponentiated inverse Rayleigh (TKEIR) distributions[12]. The MLEs and their standard errors for KM, (IPM), and (TKEIR) distributions, along with statistics criteria like, -Maximized Loglikelihood (−L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion $(CAIC)$, Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion (HQIC) are computed for our data and displayed in Tables 2 The best model is the one that acquires the lowest values for the information criteria.

Table 2. The estimates and the standard errors (in parentheses) and goodness-of-fit statistics for bladder cancer patients' data

Model	α	a	b	θ	β	λ	-L	AIC	CAIC	BIC	HQIC
KM	**(26.886)**	**10.017**	**(0.011)**	**-**	**-**		**(32.612)**	**(75.224)**	**(76.277)**	**(85.94)**	**(79.438)**
IPM	1.459	-	-	0.64	0.134		119.932	249.863	250.916	260.579	254.078
TKEIR	2.638	2.638	13.803	2.637	-	0.563	195.701	401.402	402.455	412.118	405.617

6. Conclusion

In this article, we introduce a novel model termed the Kumaraswamy Muth distribution (KM), which serves as a generalization of the Muth distribution originally proposed by Muth in 1977. We explore various mathematical properties of this new model and also derive order statistics functions. To demonstrate the versatility of the KM distribution, we illustrate the shapes of its probability density function (pdf), cumulative distribution function (cdf), and hazard function.

Furthermore, we employ the maximum likelihood method to estimate the model parameters. We then apply the KM distribution to two distinct real datasets, showcasing its practical utility. The results from these applications indicate that the KM distribution exhibits greater flexibility when compared to related models.

References

1. Jodra, P.; Jiménez-Gamero, M.D.; Alba-Fernández, M.V. On the Muth Distribution. *Math. Model. Anal.* **2015**, *20*, 291–310, doi:10.3846/13926292.2015.1048540.

2. Al-Babtain, A.A.; Elbatal, I.; Chesneau, C.; Jamal, F. The Transmuted Muth Generated Class of Distributions with Applications. *Symmetry (Basel).* **2020**, *12*, 1–18, doi:10.3390/sym12101677.

3. Jones, M.C. Kumaraswamy's Distribution: A Beta-Type Distribution with Some Tractability Advantages. *Stat. Methodol.* **2009**, *6*, 70–81, doi:10.1016/j.stamet.2008.04.001.

4. Selim, M.A.; Badr, A.M. The Kumaraswamy-Generalized Power Weibull Distribution New Families of Distributions View Project The Kumaraswamy Generalized Power Weibull Distribution. **2016**, *6*.

5. Unnikrishnan Nair, N.; Sankaran, P.G.; Balakrishnan, N. *Quantile-Based Reliability Analysis*;

6. Jodrá, P. Computer Generation of Random Variables with Lindley or Poisson-Lindley Distribution via the Lambert W Function. *Math. Comput. Simul.* **2010**, *81*, 851–859, doi:10.1016/j.matcom.2010.09.006.

7. Corless, R.M.; Jeffrey, D.J. *The Lambert W Function*; 2013;

8. Ozalp, N.; Bairamov, E. Uniform Convergence and Computation of the Generalized Exponential Integrals. *J. Math. Chem.* **2011**, *49*, 520–530, doi:10.1007/s10910-010-9756-5.

9. Letters, P.; April, N.; Exponential, T.H.E.; Distribution, I.; June, R.; September, R. PTT Dr. Neher Laboratories, Leidschendam, The Netherlands. **1987**, *5*, 209–211.

10. Jennrich, R.I.; Robinson, S.M. A Newton-Raphson Algorithm for Maximum Likelihood Factor Analysis. *Psychometrika* **1969**, *34*, 111–123, doi:10.1007/BF02290176.

11. Chesneau, C.; Agiwal, V. Statistical Theory and Practice of the Inverse Power Muth Distribution. *J. Comput. Math. Data Sci.* **2021**, *1*, 100004, doi:10.1016/j.jcmds.2021.100004.

12. Transmuted Kumaraswamy Exponentiated Inverse Rayleigh Distribution ABDALLAH MAHMUOD MOHAMMED BADR. **2017**, *7*, 41–55.