Bayesian Estimation For Topp-Leone Weibull Distribution
Based on Dual Generalized Order Statistics

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Abstract

In this paper, a Topp-Leone Weibull distribution is presented as a composite distribution. Bayes estimators of the unknown parameters, reliability and hazard rate function of the Topp-Leone Weibull distribution based on dual generalized order statistics are derived. The estimators are obtained under the squared error loss function as a symmetric loss function and the linear exponential loss function as an asymmetric loss function. The results are specialized to lower record values as special case of the dual generalized order statistics. In addition, credible intervals for the model parameters are constructed. A numerical example is given to illustrate the theoretical results and an application using real data is used to demonstrate how the results can be used in practice.

***Keywords*:** Topp-Leone distribution; Topp-Leone Weibull distribution, Linear exponential loss function; Dual generalized order statistics; Lower record values; Bayesian estimation.

1. Introduction

Topp-Leone (TL) distribution was proposed by Topp and Leone (1955); as an alternate model failure data. It is a continuous unimodal distribution with bounded support; therefore it is appropriate for modeling lifetime of distributions with finite support such as limited power supply, maintenance/repair resource, or design life of the system.

The *probability density function* (pdf) of Topp-Leone distribution is given by

$$f(x;\theta,b)=\frac{2\alpha}{b}\left(1-\frac{x}{b}\right)\left(2\frac{x}{b}-\left(\frac{x}{b}\right)^2\right)^{\alpha-1}, \qquad 0<x<b, \alpha>0. \tag{1}$$

The *cumulative distribution function* (cdf) of Topp-Leone distribution is as follows:

$$F(x;\theta,b)=\begin{cases} 0, & x<0,\\ \left(2\frac{x}{b}-\left(\frac{x}{b}\right)^2\right)^{\alpha}, & 0<x<b,\\ 1, & x>0, \end{cases} \tag{2}$$

where α is a shape parameter and b is a scale parameter, if α is restricted to be in (0,1), then the distribution function in (1) is J-shaped distribution.

Nadarajah and Kotz (2003) showed that TL distribution have bathtub failure rate function with widespread applications in reliability. Some attractive reliability properties were provided by Ghitany *et al.* (2005), such as the bathtub-shape hazard rate, decreasing reversed hazard rate, upside-down mean residual life, increasing expected inactivity time. Also, Zghoul (2010) studied order statistics from TL distribution and provided expressions for moments of ordered statistics from TL distribution.

Feroze and Aslam (2013) derived Bayesian estimation and prediction using a couple of non-informative priors under complete and Type II censored samples. Sindhu *et al.* (2013) obtained Bayes estimators for the shape parameter and credible intervals based on trimmed samples using different priors.

Maximum likelihood (ML) and Bayesian estimation of the parameters of TL distribution, based on lower record values under symmetric loss function were obtained by Li (2016). Also, he derived the empirical Bayes estimators. Bayesian estimation of the shape parameter, under simple and mixture priors and different loss functions, is presented by Sultan and Ahmad (2017). [For more details on TL distribution see, Zghoul (2011), Genç (2012), Khan and Khan (2015) and Bayoud (2015)].

Aryal *et al.* (2016) introduced the TL generated Weibull distribution and derived some structural properties, also they used the ML method to obtain the estimators of the parameters.

The concept of *generalized order statistics* (**gos**) was established by Kamps (1995 a, b) to unify several concepts that have been used in statistics such as ordinary order statistics, *record values* (Rv), sequential order statistics, progressive Type II censored data, Pfeifer's record model and others. The concept includes almost the models of ordered random variables which are arranged in ascending order of magnitude.

Pawlas and Szynal (2001) introduced the concept of *Dual Generalized Order Statistics* **(dgos)**, which includes the order random variables arranged in decreasing order of magnitude. Burkschat *et al.* (2003) extensively studied and discussed **dgos** to enable a common approach to descending order random variables as reversed order statistics, lower records and lower Pfeifer records. They also established the connection between **gos** and **dgos** as follows:

If $X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)$ be the **gos** based on continuous cdf G and $X_d(1,n,\tilde{m},k),\dots,X_d(n,n,\tilde{m},k)$ be the **dgos** based on a continuous cdf F.

Then

$$(F(X_d(r,n,\tilde{m},k)))\overset{d}{=}(1-G(X(r,n,\tilde{m},k))), \qquad 1\le r\le n.$$

Therefore, **gos** is used when *(1-F(x))* is in closed form and **dgos** is used when *F(x)* is in closed form.

Let the random variables, $X(1,n,m,k)$, $X(2,n,m,k),\dots,X(n,n,m,k)$ be n **dgos** from an absolutely cdf, *F(x)*, and pdf, *f(x)*, then their joint pdf has the form

$$f^{X(1,n,m,k),\dots,X(n,nm,k)}(x_1,\dots,x_n)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}(F(x_i))^m f(x_i)\right](F(x_n))^{k-1}f(x_n), \qquad (3)$$

where $F^{-1}(1)\ge x_1\dots\ge x_n\ge F^{-1}(0)$,

and

$n\in\mathbb{N}, k\ge 1, m_1,\dots,m_{n-1}=m, m\in\mathbb{R}$ be the parameters such that

$\gamma_r=k+(n-r)(m+1)\ge 1,\ for\ all\ 1\le r\le n.$

The marginal pdf of r th **dgos** $X\,(r,n,m,k), 1\le r\le n$ is given by; [See Khan and Khan (2015)]

$$f^{(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r-1}f(x)\,g_m^{r-1}(F(x)), \qquad (4)$$

and the joint pdf of $X\,(r,n,m,k)=x$ and $X\,(s,n,m,k)=y$, $-\infty\le x\le y\le\infty$, is given by

$$f^{X(r,n,m,k),\dots,X(s,nm,k)}(x,y)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}(F(x))^m f(x) g_m^{r-1}F(x)$$

$$\times\left[[h_m(F(y))-h_m(F(x))]^{s-r-1}(F(y))^{\gamma_s-1}F(y),\right. \qquad (5)$$

where $C_{r-1}=\prod_{j=1}^{r}\gamma_j$, $g_m(x)=h_m(x)-h_m(1)$, $X\in[0,1)$,

$$h_m(x)=\begin{cases}-\frac{1}{m+1}x^{m+1}, & m\ne 1\\ -\log x, & m=1\end{cases}. \qquad (6)$$

Lower records are of great interest and importance in many areas of real life applications involving data relating to weather, sports (athletic events), industry, economics, biomedical sciences, engineering, the environmental sciences, actuarial sciences, management sciences, social sciences and life testing; for example the lowest stock markets figure, also they are useful in reliability theory, meteorology, hydrology, seismology and mining . [See Khan and Faizan (2014)].

The joint pdf of the first n lower record values $X_{d(1)}, X_{d(2)}, \ldots, X_{d(n)}$ when $m = -1$ and $k = 1$ in (3) is as follows:

$$f^{X_{d(1)},\ldots,X_{d(n)}}(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n-1} \frac{f(x_i)}{F(x_i)}\right) f(x_n), \qquad X_n > X_{n-1}, \ldots > X_1 . \qquad (7)$$

The rest of this paper is organized as follows: Section 2 presents Topp-Leon Weibull distribution as a composite distribution. Bayesian estimation based on **dgos**; under squared error and linear exponential loss functions, is discussed in Section 3. Also, Bayes estimators for the unknown parameters, *reliability function* (rf) and *hazard rate function* (hrf) are derived based on lower records, in Section 4. In Section 5, a numerical example, through simulated and real data, are given to illustrate the theoretical results.

2. Topp-Leone Weibull Distribution

Considering b=1 in (2); without any loss of generality, a random variable X is distributed as the TL distribution with parameter α denoted by $X \sim TL(\alpha)$ with a cdf

$$H_{TL}(x) \equiv H_{TL}(x; \alpha) = (2x - x^2)^{\alpha}, \qquad 0 < x < 1, \alpha > 0. \qquad (8)$$

The corresponding pdf is

$$h_{TL}(x; \alpha) = 2\,\alpha\,(1 - x)(2x - x^2)^{\alpha - 1}, \; 0 < x < 1, \alpha > 0 . \qquad (9)$$

The rf and the hrf are, respectively, given by

$$R_{TL}(x; \alpha) = 1 - (2x - x^2)^{\alpha}, \qquad (10)$$

and

$$hr_{TL}(x;\alpha) = \frac{2\alpha(1-x)(2x-x^2)^{\alpha-1}}{1-(2x-x^2)^{\alpha}}. \quad (11)$$

On composition of distribution functions, see AL-Hussaini (2012). A composition of H, given by (8) and a cdf G, with positive support, yields a new cdf, given below

$$F(t) = H(G(t)) = (2G(t) - (G(t))^2)^{\alpha}, \quad (12)$$

In particular, if G is Weibull distribution; denoted by $W \sim (\lambda, \vartheta)$, with cdf as

$$G(t) \equiv G(t;\lambda,\vartheta) = 1 - \exp(-(\lambda t)^{\vartheta}), t > 0, \lambda, \vartheta > 0 \quad (13)$$

Substituting (13) in (12), the cdf for *Topp-Leone Weibull distribution* ($TLW(\alpha,\lambda,\vartheta)$) is given by

$$F_{TLW}(t) \equiv F_{TLW}(t;\alpha,\lambda,\vartheta) = (1-\exp(-2(\lambda t)^{\vartheta}))^{\alpha},\ t > 0,\ \alpha,\lambda,\vartheta > 0. \quad (14)$$

The pdf, corresponding to the cdf given in (14), is as follows:

$$f_{TLW}(t;\alpha,\lambda,\vartheta) = 2\,\alpha\,\vartheta\,\lambda\,(\lambda t)^{\vartheta-1}\exp(-2(\lambda t)^{\vartheta})(1-\exp(-2(\lambda t)^{\vartheta}))^{\alpha-1},\quad t > 0,\ \alpha,\lambda,\vartheta > 0, \quad (15)$$

where α, ϑ are shape parameters and λ is a scale parameter.

Figures 1 and 2, describe the PDF of TLW distribution for different parameter values.

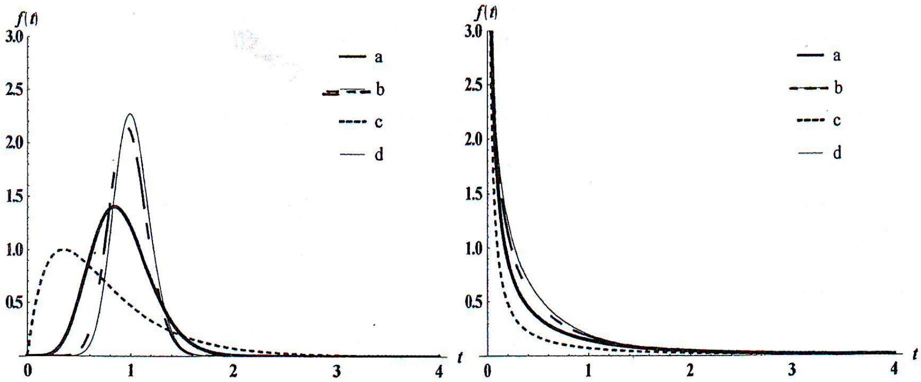

Figure 1 **Figure 2**

Plots of the probability density function of TLW distribution

In Figure 1, the TLW density is monotonically decreasing at $= 0.5, 0.7, 0.8, 0.9,\ \theta = 0.4, 0.6, 0.75, 0.8,\ \lambda = 0.6, 0.7, 0.9, 1,$ while in Figure 2 it has unimodal, approximately symmetric and negative skewed curves at $\alpha = 2, 3, 4, 5, \theta = 1, 2, 3, 5\ , \lambda = 1, 1, 1, 1.$

The rf and hrf for TLW distribution are, respectively, as :

$$R_{TLW}(t; \alpha, \lambda, \vartheta) = 1 - \left(1 - \exp\left(-2(\lambda t)^{\vartheta}\right)\right)^{\alpha}, \qquad t > 0,\ \alpha, \lambda, \vartheta > 0, \tag{16}$$

and

$$h_{TLW}(t; \alpha, \lambda, \vartheta) = \frac{2\,\alpha\,\vartheta\,\lambda\,(\lambda t)^{\vartheta-1}\exp(-2(\lambda t)^{\vartheta})(1-\exp(-2(\lambda t)^{\vartheta}))^{\alpha-1}}{1-(1-\exp(-2(\lambda t)^{\vartheta}))^{\alpha}}, \qquad t > 0,\ \alpha, \lambda, \vartheta > 0, \tag{17}$$

Figures 3 and 4, show the hrf of TLW for different parameter values.

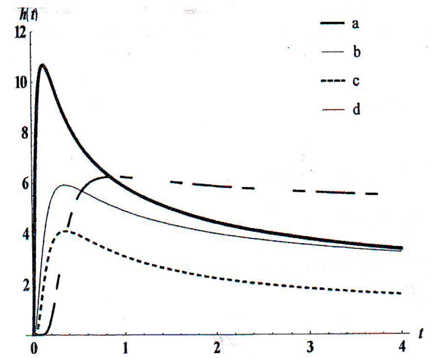

Figure 3

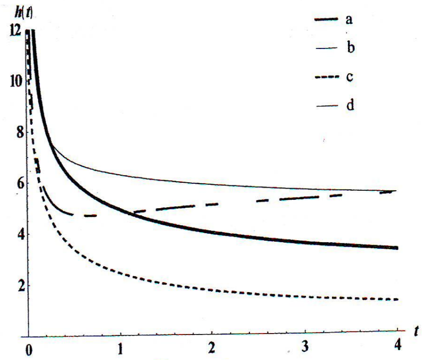

Figure 4

The hrf's of Topp-Leone Weibull distribution for different parameter values

Figures 3 and 4 show that the hrf of TLW for different parameter values is quite flexible for modeling survival data. The hrf is increasing then decreasing at $\alpha = 38, 12, 16, 20,\ \theta = 0.5, 0.7, 0.9, 1.1,\ \lambda = 26, 8, 4, 2$ and at $\alpha = 2, 0.001, 0.001, 0.001, \theta = 0.5, 0.7, 0.9, 1.1,\ \lambda = 6, 6, 4, 2.$ The TLW represents most major hazard shapes.

3. Bayesian Estimation Based On Dual Generalized Order Statistics

In this section, Bayesian approach is used to estimate the parameters, rf and hrf of TLW distribution based on **dgos;** under *squared error loss* (SEL) function and *Linear exponential* (LINEX) loss function, using informative prior. Also the credible intervals for the parameters, rf and hrf are obtained.

3.1 Bayesian estimation under squared error loss function

Bayes estimators of the parameters, rf and hrf of the TLW distribution are considered under SEL function as a symmetric loss function.

Suppose that *T (1, n, m, k), T (2, n, m, k), …, T(n, n, m, k)* be *n* **dgos** from TLW distribution, the likelihood function can be obtained by substituting (14) and (15) in (4) and written as

$$L(\underline{\Theta}|\underline{t}) \propto \prod_{i=1}^{n-1}((1-\exp(-2(\lambda t_i)^{\vartheta}))^{m\alpha}\ \alpha\,\vartheta\lambda(\lambda t_i)^{\vartheta-1}\exp(-2(\lambda t_i)^{\vartheta})$$
$$\times(1-\exp(-2(\lambda t_i)^{\vartheta}))^{\alpha-1})$$
$$\times((1-\exp(-2(\lambda t_n)^{\vartheta}))^{\alpha(k-1)}\ \alpha\,\vartheta\lambda(\lambda t_n)^{\vartheta-1}\exp(-2(\lambda t_n)^{\vartheta})$$
$$\times(1-\exp(-2(\lambda t_n)^{\vartheta}))^{\alpha-1})$$
$$=\alpha^n\,\vartheta^n\lambda^{n\vartheta}\prod_{i=1}^{n}t_i^{\vartheta-1}\exp(-2(\lambda t_i)^{\vartheta})\,(1-\exp(-2(\lambda t_i)^{\vartheta}))^{\alpha-1}$$
$$\times\left[\prod_{i=1}^{n-1}(1-\exp(-2(\lambda t_i)^{\vartheta}))^{m\alpha}\right]\left[(1-\exp(-2(\lambda t_n)^{\vartheta}))^{\alpha(k-1)}\right]. \qquad (20)$$

Considering the prior knowledge of the vector of parameters $\underline{\Theta}$, is adequately represented by conjugate prior which is the gamma distribution with the hyper parameters a_j, b_j and pdf as below

$$g(\theta_j; a_j, b_j)=\frac{b_j^{a_j}}{\Gamma(a_j)}\theta_j^{a_j-1}\,e^{-b_j\theta_j}, \qquad \theta_j, a_j, b_j>0,\ j=1,2,3.$$

where $\theta_1=\alpha,\ \theta_2=\lambda$ and $\theta_3=\vartheta$, which are independent.

Then the joint prior distribution of the unknown parameters has a joint pdf given by

$$\pi(\theta_j; a_j, b_j) = \prod_{j=1}^{3}\left[\frac{b_j^{a_j}}{\Gamma(a_j)}\theta_j^{a_j-1}\, e^{-b_j\theta_j}\right], \qquad \theta_j > 0; a_j, b_j > 0\,. \tag{20}$$

Hence, the joint posterior distribution can be derived using (20) and (21) as follows:

$$\pi(\underline{\Theta} \mid \underline{t}) \propto L(\underline{\Theta}|\underline{t})\, \pi(\theta_j; a_j, b_j)$$

$$\propto \alpha^{n+a_1-1}\, \vartheta^{n+a_2-1}\lambda^{n\vartheta+a_3-1}$$

$$\times\left[\prod_{i=1}^{n} t_i^{\vartheta-1}\exp(-2(\lambda t_i)^{\vartheta} - b_1\alpha - b_2\vartheta - b_3\lambda)(1-\exp(-2(\lambda t_i)^{\vartheta}))^{\alpha-1}\right]$$

$$\times\left[\prod_{i=1}^{n-1}(1-\exp(-2(\lambda t_i)^{\vartheta}))^{m\alpha}\right]\left[(1-\exp(-2(\lambda t_n)^{\vartheta}))^{\alpha(k-1)}\right]. \tag{22}$$

The marginal posteriors of α, λ and ϑ can be obtained by integrating the joint posterior distribution given by (22) with respect to the other parameters, that is the marginal posterior density is given by

$$\pi_j^*(\underline{\Theta} \mid \underline{t}) = \int\int_{\theta_i} \pi(\underline{\Theta} \mid \underline{t})\, d\theta_i\,, \qquad i,j = 1,2,3,\ i \neq j \qquad \theta_j > 0. \tag{23}$$

Under SEL function the Bayes estimators; which are the mean of the posterior distribution, can be derived as follows:

$$\theta_{j(SE)}^* = E(\theta_j|\,\underline{t}) = \int_{\underline{\theta}} \theta_j\, \pi_j^*(\underline{\Theta} \mid \underline{t})\, d\theta_j\,, \qquad j = 1,2,3, \qquad \theta_j > 0\,, \tag{24}$$

where $\theta_1 = \alpha,\ \theta_2 = \lambda$ and $\theta_3 = \vartheta$.

The Bayes estimators of the rf and the hrf under SEL function; which are the posterior expectations, can be obtained as follows:

$$R_{(SE)}^*(t) = E(R(t)|\underline{t}) = \int_{\underline{\Theta}} R(t)\,\pi(\underline{\Theta} \mid \underline{t})\, d\underline{\Theta}$$

$$= \int_{\underline{\Theta}} (1-(1-\exp(-2(\lambda t_n)^{\vartheta}))^{\alpha})\, \alpha^{n+a_1-1}\, \vartheta^{n+a_2-1}\lambda^{n\vartheta-a_3-1}$$

$$\times\left[\prod_{i=1}^{n} t_i^{\vartheta-1}\exp(-2(\lambda t_i)^{\vartheta} - b_1\alpha - b_2\vartheta - b_3\lambda)(1-\exp(-2(\lambda t_i)^{\vartheta}))^{\alpha-1}\right]$$

$$\times\left[\prod_{i=1}^{n-1}(1-\exp(-2(\lambda t_i)^{\vartheta}))^{m\alpha}\right]\left[(1-\exp(-2(\lambda t_n)^{\vartheta}))^{\alpha(k-1)}\right] d\underline{\Theta}\,, \tag{25}$$

and

$$h^*_{(SE)}(t) = E(h(t)|\underline{t}) = \int_{\underline{\Theta}} h(t)\, \pi(\underline{\Theta} \mid \underline{t})\, d\underline{\Theta}$$

$$= \int_{\underline{\Theta}} \frac{2\,\alpha\,\vartheta\,\lambda\,(\lambda t_n)^{\vartheta-1}\exp(-2(\lambda t_n)^{\vartheta})(1-\exp(-2(\lambda t_n)^{\vartheta}))^{\alpha-1}}{1-(1-\exp(-2(\lambda t_n)^{\vartheta}))^{\alpha}}$$

$$\times\left[\prod_{i=1}^{n} t_i^{\vartheta-1}\exp(-2(\lambda t_i)^{\vartheta} - b_1\alpha - b_2\vartheta - b_3\lambda)(1-\exp(-2(\lambda t_i)^{\vartheta}))^{\alpha-1}\right]$$

$$\times\left[\prod_{i=1}^{n-1}(1-\exp(-2(\lambda t_i)^{\vartheta}))^{m\alpha}\right]\left[(1-\exp(-2(\lambda t_n)^{\vartheta}))^{\alpha(k-1)}\right] d\underline{\Theta}\,. \qquad (26)$$

Equations (24-26) can be solved numerically to obtain the Bayes estimates of the parameters, rf and hrf of the TLW distribution based on SEL function.

3.2 Bayesian estimation under linear exponential loss function

Although SEL function is the most loss function used in literature and its symmetric nature gives equal weight to over and under estimation of the parameters, in life testing over estimation is more serious than under estimation or vice versa.

Varian (1975) introduced the LINEX loss function to be of the form

$$\zeta(\Delta) = b_0[e^{c\Delta} - c\Delta - 1],$$

where $c \neq 0, b_0 > 0$, and $\Delta = \hat{u}(\Theta) - u(\Theta)$.

AL-Hussaini and Hussein (2011) suggested using the *squared exponential* (SQUAREX) loss function with the form

$$\zeta^*(\Delta) = b_0[e^{c\Delta} - d\Delta^2 - c\Delta - 1],$$

where $d \neq 0, b_0, c$ *and* Δ are as before.

The SQUAREX loss function reduces to the LINEX loss function if $d = 0$, but if $c = 0$, the SQUAREX loss function reduces to SEL function.

The LINEX loss function will be applied to estimate the parameters, rf and hrf of TLW distribution based on **dgos**, then the Bayes estimators of $u(\underline{\Theta})$, a function of the vector of the parameters Θ, is given by

$$\hat{u}_L(\underline{\Theta}) = -\frac{1}{c}\ln E(e^{-u(\underline{\Theta})}|\underline{t}) = -\frac{1}{c}\ln \int_{\underline{\Theta}} e^{-u(\underline{\Theta})}\,\pi(\underline{\Theta}|\underline{t})d\underline{\Theta}, \qquad (27)$$

where $\pi(\underline{\Theta}|\underline{t})$ is the posterior pdf of the vector of the parameters $\underline{\Theta}$, given the data $\underline{t}$,

$\underline{\Theta} = \alpha, \lambda, \vartheta\, , R_{TLW}(t) or\ h_{TLW}(t)$ and $d\underline{\Theta} = d\alpha\, d\lambda\, d\vartheta.$ (28)

In parallel with the steps used for derivation of the Bayes estimators of the parameters, rf and hrf under SEL function, the Bayes estimators of $\underline{\Theta} = (\theta_1, \theta_2, \theta_3)'$, rf and hrf under the LINEX loss function, based on **dgos** are given, respectively, by

$$\hat{\theta}_{j(LNX)} = \frac{-1}{c}\ln[E(e^{-c\theta_j}|\underline{t})] = \frac{-1}{c}\ln[\int_0^\infty e^{-c\theta_j}.\pi_j^*(\underline{\Theta} \mid \underline{t})\, d\theta_j], \quad j = 1,2,3, \quad (29)$$

$$\hat{R}_{LNX}(t) = \frac{-1}{c}\ln[E(e^{-cR(t)}|t)] = \frac{-1}{c}\ln\left[\int_{\underline{\Theta}} e^{-cR(t)}.\pi(\underline{\Theta} \mid \underline{t})\, d\underline{\Theta}\right], \quad (30)$$

and

$$\hat{h}_{LNX}(t) = \frac{-1}{c}\ln[E(e^{-ch(t)}|t)] = \frac{-1}{c}\ln\left[\int_{\underline{\Theta}} e^{-ch(t)}.\pi(\underline{\Theta} \mid \underline{t})\, d\underline{\Theta}\right], \quad (31)$$

where $\theta_1 = \alpha,\ \theta_2 = \lambda$ and $\theta_3 = \vartheta$ and $\int_{\underline{\Theta}} = \int_\alpha^\infty \int_\lambda^\infty \int_\vartheta^\infty$ and $c \neq 0.$

• **Credible intervals based on dual generalized order statistics**

The Bayesian analog to the confidence interval is called a credibility interval. In general, $(L(\underline{t}), U(\underline{t}))$ is $100(1 - \omega)\%$ credibility interval of $\underline{\Theta}$ if

$$P[L(\underline{t}) < \underline{\Theta} < U(\underline{t})|\underline{t}] = \int_{L(\underline{t})}^{U(\underline{t})} \pi^*(\underline{\Theta}|\underline{t})\, d\underline{\Theta} = 1 - \omega. \quad (32)$$

Using the marginal posterior distribution given by (23), then a $100(1 - \omega)\%$ credibility interval for θ_j based on **dgos** is $\left(L_j(\underline{t}), U_j(\underline{t})\right)$, where

$$P[\theta_j > L_j(\underline{t})|\underline{t}] = \int_{L_j(\underline{t})}^\infty \pi_j^*(\underline{\Theta} \mid \underline{t})\, d\theta_j = 1 - \frac{\omega}{2}, \quad j = 1,2,3, \quad (33)$$

and

$$P[\theta_j > U_j(\underline{t})|\underline{t}] = \int_{U_j(\underline{t})}^\infty \pi_j^*(\underline{\Theta} \mid \underline{t})\, d\theta_j = \frac{\omega}{2}, \quad j = 1,2,3.$$

4. Bayesian Estimation Based on Lower Record Values

The lower record values can be obtained from **dgos** as a special case by taking $m_i = -1$, $\forall\, i = 1, \ldots, r, \gamma_r = 1$ and $k = 1$. Bayes estimators of the parameters, rf and hrf based on lower record values from TLW distribution are derived under the SEL function and LINEX loss function. Also the credible intervals of the parameters are considered.

The joint posterior distribution based on lower records is as follows:

$$\pi^{**}(\underline{\Theta} \mid \underline{t}) = \alpha^{n+a_1-1}\, \vartheta^{n+a_2-1} \lambda^{n\vartheta-a_3-1} \left[\prod_{i=1}^{n-1}(1-\exp(-2(\lambda t_i)^{\vartheta}))^{-\alpha}\right]$$
$$\times \left[\prod_{i=1}^{n} t_i^{\vartheta-1} \exp(-2(\lambda t_i)^{\vartheta} - b_1\alpha - b_2\vartheta - b_3\lambda)(1-\exp(-2(\lambda t_i)^{\vartheta}))^{\alpha-1}\right], \tag{34}$$

The marginal posteriors of α, λ and ϑ can be obtained by integrating the joint posterior distribution given by (34) with respect to the other parameters, that is the marginal posterior density is given by

$$\pi_j^{**}(\underline{\Theta} \mid \underline{t}) = \iint_{\theta_i} \pi^{**}(\underline{\Theta} \mid \underline{t})\, d\theta_i\,, \qquad i,j = 1,2,3, \qquad i \neq j, \quad \theta_j > 0. \tag{36}$$

Under SEL function the Bayes estimators based on lower records can be derived as follows:

$$\hat{\theta}_{j(SE)} = E(\theta_j \mid \underline{t}) = \int_{\underline{\theta}} \theta_j\, \pi_j^{**}(\underline{\Theta} \mid \underline{t})\, d\theta_j\,, \qquad j = 1,2,3, \quad \theta_j > 0\,, \tag{37}$$

where $\theta_1 = \alpha,\ \theta_2 = \lambda$ and $\theta_3 = \vartheta$.

Under LINEX loss function, the Bayes estimators of $\underline{\Theta} = (\theta_1, \theta_2, \theta_3)'$, rf and hrf based on lower record values are given, respectively, by

$$\hat{\theta}_{j(LNX)} = \frac{-1}{c}\ln[E(e^{-c\theta_j}|\underline{t})] = \frac{-1}{c}\ln[\int_0^{\infty} e^{-c\theta_j}.\pi_j^{**}(\underline{\Theta} \mid \underline{t})\, d\theta_j], \qquad j = 1,2,3, \tag{38}$$

$$\hat{R}_{(LNX)}(t) = \frac{-1}{c}\ln[E(e^{-cR(t)}|\underline{t})] = \frac{-1}{v}\ln\left[\int_{\underline{\Theta}} e^{-cR(t)}.\pi^{**}(\underline{\Theta} \mid \underline{t})\, d\underline{\Theta}\right], \tag{39}$$

and

$$\hat{h}_{(LNX)}(t) = \frac{-1}{c}\ln[E(e^{-ch(t)}|\underline{t})] = \frac{-1}{c}\ln\left[\int_{\underline{\Theta}} e^{-ch(t)}.\pi^{**}(\underline{\Theta} \mid \underline{t})\, d\underline{\Theta}\right], \tag{40}$$

where $\int_{\underline{\Theta}} = \int_{\alpha}^{\infty}\int_{\lambda}^{\infty}\int_{\vartheta}^{\infty}$ and $c \neq 0$.

- **Credible intervals based on Lower records**

A $100(1-\omega)\%$ credibility interval for θ_j based on lower record values is $\left(L_{lj}(\underline{t}), U_{lj}(\underline{t})\right)$ can be obtained through using steps analogous to those used for obtaining credible intervals based on **dgos**.

In general, $(L_l(\underline{t}), U_l(\underline{t}))$ is $100(1-\omega)\%$ credibility interval of $\underline{\Theta}$ if

$$P[L_l(\underline{t}) < \underline{\Theta} < U_l(\underline{t})|\underline{t}] = \int_{L_l(\underline{t})}^{U_l(\underline{t})} \pi^{**}(\underline{\Theta}|\underline{t})\, d\underline{\Theta} = 1 - \omega. \quad (41)$$

Using the marginal posterior distribution ~~given by~~ (31), then a $100(1 - \omega)\%$ credibility interval for θ_j based on lower record values is $\left(L_{lj}(\underline{t}), U_{lj}(\underline{t})\right)$, where

$$P[\theta_j > L_{lj}(\underline{t})|\underline{t}] = \int_{L_{lj}(\underline{t})}^{\infty} \pi_j^{**}(\underline{\Theta} \mid \underline{t})\, d\theta_j = 1 - \frac{\omega}{2}, \quad j = 1,2,3, \quad (42)$$

and

$$P[\theta_j > U_{lj}(\underline{t})|\underline{t}] = \int_{U_{lj}(\underline{t})}^{\infty} \pi_j^{**}(\underline{\Theta} \mid \underline{t})\, d\theta_j = \frac{\omega}{2}, \quad j = 1,2,3. \quad (43)$$

To obtain the Bayes estimates of the parameters, rf, hrf and the credible intervals; Equations (38)-(43) should be solved numerically.

Special cases:

- Note that the results obtained in this paper for the TLW distribution give corresponding results for the TL exponential distribution, if $\vartheta=1$.
- Results obtained in this paper for the TLW distribution give corresponding results for the TL Rayleigh distribution, if $\vartheta=2$.

5. Numerical Illustration

This section aims to investigate the precision of the theoretical results of Bayesian estimation under SE and LINEX loss functions, based on lower record values through simulated and real data.

5.1 Simulation

In this subsection, Monte Carlo simulation study is conducted to illustrate the performance of the presented Bayes estimates on the basis of generated data from the TLW distribution. Bayes averages of the parameters, rf and hrf based on lower record values are computed. Moreover, credible intervals of the parameters are calculated, all the results are obtained using R programming language.

a. For given values of α, λ and ϑ random samples of size n are generated from TLW distribution observing that if U is uniform distribution (0,1), then

$$t = \frac{1}{\lambda}\left[\log\left(1 - u^{\frac{1}{\alpha}}\right)^{-\frac{1}{2}}\right]^{\frac{1}{\vartheta}}, \tag{44}$$

b. For each sample size n, consider the first observation is the first lower record value t_1 denote it R_1, which is considered as the maximum and the second observation t_2 denote it R_2 which is smaller than the maximum $(t_1 > t_2)$ record and if $t_1 \leq t_2$ ignore it and repeat until you get the sample of Rv_records.

c. For the number of the Rv records, the initial values of the parameters α, λ and ϑ and the hyper parameters of the prior distribution, the Bayes averages of the parameters, rf and hrf under SE and LINEX loss functions are computed. The computations are performed using R packages.

d. Table 1 in the appendix presents the Bayes averages using SEL function and gamma prior, their biases, *relative errors* (RE) $= \frac{\sqrt{mean\ square\ error}}{population\ paramter}$ and the lengths of 95% credible interval for the parameters with the true values $\alpha = 4.8$, $\vartheta = 0.54$ and $\lambda = 0.05$, based on lower records Rv=(5, 7, 10) and replications NR = 10000.

e. Table 2 in the appendix displays the Bayes averages using LINEX loss function and gamma prior, their biases, REs and the lengths of 95% credible interval for the parameters with the true values $\alpha = 4.8$, $\vartheta = 0.54$ and $\lambda = 0.05$, based on lower records Rv=(5, 7, 10), $c = (-0.1, 0.01, 0.1)$ and number of replications NR = 10000.

5.2 Real data

The main aim of this subsection is to demonstrate how the theoretical results can be used in practice. The data set used in this application consists of the waiting times between 65 consecutive eruptions of the Kiama Blowhole. The Kiama Blowhole is a touristic attraction located nearly 120 km to the south of Sydney. The swelling of the ocean pushes the water through a hole bellow a cliff. The water then erupts through an exit usually drenching whoever is nearby. The time between eruptions on July 12, 1998 were recorded using a digital watch by Professor Jim Irish and were obtained from StatSci.org in June 28, 2012 at http://www.statsci.org/data/oz/kiama.html.

The actual data are:

83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28,56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

The Kolmogorov-Smirnov goodness of fit test is applied to check the validity of the fit of the model. The p value is 0.2106, shows that the model fits the data very well.

a. Table 3 in the appendix presents the Bayes estimates using SEL function and gamma prior, their biases and REs, based on lower records Rv=5 .
b. Table 4 in the appendix displays the estimates using LINEX loss function and gamma prior, their biases and REs, based on lower records Rv=5 and $c = (-0.1, 0.01, 0.1)$.

5.3 Concluding Remarks

- From Tables 1 and 2, it is clear that the Bayes estimates are consistent and very close to the true parameter values as the sample size (number of lower Rv) increases. This is easily observed from comparing the Biases and REs. Also, the lengths of the credible intervals become narrower as the sample of lower Rv increases.
- The previous remark is expected since increasing the sample size means that more information is provided by the sample and hence increases the accuracy of the estimates.
- From Tables 1 and 2, one can notice that the Biases and REs for the estimates of the parameters, rf and hrf under LINEX loss function have less values than the corresponding Biases and REs of the estimates under SEL function.
- The constant c determines the trend of the loss function. When overestimation is more serious than under estimation, positive values of c are used while negative values of c is used in reverse situations. For small value of $|c|$, the loss is almost symmetric and $\zeta(\Delta)$ behaves similar to the squared error loss function.
- The results based on the real data ensure the Monte Carlo simulation results.

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Appendix

Table 1. Average estimates, biases, relative error and the length of 95% credible interval for the parameters α, ϑ *and* λ based on lower record values under the SEL function ($N = 10000, \alpha = 4.8$, $\vartheta = 0.54$ and $\lambda = 0.05$)

Rv	Θ	Average	Bias	RE	Length
5	α	4.79766	0.00174	4.15356 e-07	0.00370
	ϑ	0.53616	0.00094	2.24607 e-06	0.00431
	λ	0.04840	0.00190	4.21608 e-05	0.00332
	$R(x_0)$	0.06325	0.00208	3.92559e-06	0.00391
	$h(x_0)$	0.30784	0.00144	4.61681e-06	0.00277
7	α	4.79776	0.00126	3.14920 e-07	0.00352
	ϑ	0.53709	0.00070	4.20856 e-07	0.00359
	λ	0.05104	0.00073	2.20439 e-05	0.00309
	$R(x_0)$	0.06130	0.00017	2.18936 e-06	0.00269
	$h(x_0)$	0.30844	0.00095	3.44067e-06	0.00253
10	α	4.79814	0.00120	2.14230 e-07	0.00254
	ϑ	0.53814	0.00058	2.35586 e-07	0.00248
	λ	0.05043	0.00062	1.47425 e-05	0.00185
	$R(x_0)$	0.06100	0.00013	1.79785e-06	0.00199
	$h(x_0)$	0.30893	0.00035	1.00547e-06	0.00168

Table 3 Estimates, biases, relative errors and the based on lower record values under the SEL function and real data

Rv	$\underline{\Theta}$	Estimate	Bias	RE
5	α	4.79707	0.00067	7.72093e-08
	ϑ	0.53809	0.00099	2.23857e-06
	λ	0.05061	0.00031	4.27628e-06
	$R(x_0)$	0.06150	0.00033	4.997019e-06
	$h(x_0)$	0.30353	0.00396	3.106216e-05

Table 4. Estimates, biases and relative errors based on lower record values under the LINEX loss function and real data

Rv	$\underline{\Theta}$	$c = -0.1$			$c = 0.01$			$c = 0.1$		
		Estimate	Bias	RE	Estimate	Bias	RE	Estimate	Bias	RE
5	α	4.79570	0.00142	2.85423 e-07	4.79628	0.00139	2.75110e-07	4.79718	0.00110	2.60017e-07
	ϑ	0.53655	0.00089	2.24589 e-06	0.53723	0.00078	2.24430e-06	0.5359	0.00046	2.04268e-06
	λ	0.04938	0.00192	4.12454 e-05	0.05150	0.00186	4.12098e-05	0.05056	0.00127	4.00239e-05
	$R(x_0)$	0.06268	0.00078	3.22590 e-06	0.06186	0.00052	3.12560e-06	0.05961	0.00046	3.10189e-06
	$h(x_0)$	0.30617	0.00134	4.56190 e-06	0.30918	0.00120	4.45237e-06	0.30588	0.00112	4.30105e-06

Table 2. Averages, biases, relative errors and the lengths of 95% credible interval for the parameters α, ϑ *and* λ based on lower record values under the LINEX loss function ($\alpha = 4.8$, $\vartheta = 0.54$ and $\lambda = 0.05$)

Rv	Θ	$c = -0.1$				$c = 0.01$				$c = 0.1$			
		Average	Bias	RE	Length	Average	Bias	RE	Length	Average	Bias	RE	Length
5	α	4.79749	0.00146	2.85443 e-07	0.00342	4.79750	0.00010	2.75117e-07	0.00250	4.79765	0.00008	2.69117e-07	0.00385
	ϑ	0.53576	0.00093	2.24600 e-06	0.00381	0.53730	0.00079	2.24589e-06	0.00301	0.53803	0.00059	2.14398e-06	0.00361
	λ	0.05080	0.00189	4.12600 e-05	0.00535	0.05059	0.00168	4.12598e-05	0.00456	0.05146	0.00157	4.00989e-05	0.00323
	$R(x_0)$	0.06025	0.00092	3.22593 e-06	0.00335	0.06239	0.00089	3.22590e-06	0.00252	0.06000	0.00065	3.12489e-06	0.00206
	$h(x_0)$	0.30778	0.00140	4.56187 e-06	0.00398	0.30659	0.00136	4.48187e-06	0.00243	0.30927	0.00127	4.35185e-06	0.00370
7	α	4.79494	0.00112	2.26878e-07	0.00332	4.79783	0.00107	2.24920e-07	0.00242	4.79895	0.00100	1.34920e-07	0.00237
	ϑ	0.53553	0.00067	3.40586 e-07	0.00271	0.53768	0.00048	3.39486e-07	0.00251	0.53772	0.00030	2.00486e-07	0.00285
	λ	0.05058	0. 00069	2.20035 e-05	0.00421	0.04761	0. 00062	2.19935e-05	0.00298	0.04923	0. 00048	2.07893e-05	0.00273
	$R(x_0)$	0.06081	0.00036	2.17835e-06	0.00290	0.06134	0.00025	2.17829e-06	0.00210	0.06151	0.00017	2.00529e-06	0.00189
	$h(x_0)$	0.30798	0.00094	3.34176 e-06	0.00260	0.30867	0.00071	3.33987e-06	0.00236	0.30739	0.00065	2.31687e-06	0.00296
10	α	4.79752	0.00109	1.71898 e-07	0.00289	4.79802	0.00100	1.69898e-07	0.00191	4.79976	0.00009	1.20253e-07	0.00192
	ϑ	0.53582	0.00045	2.35570 e-07	0.00256	0.53713	0.00035	2.34990e-07	0.00176	0.53899	0.00020	2.00057e-07	0.00262
	λ	0.05149	0.00060	1.47314 e-05	0.00316	0.05048	0.00049	1.47310e-05	0.00159	0.05057	0.00037	1.00235e-05	0.00197
	$R(x_0)$	0.06108	0.00012	1.78765 e-06	0.00288	0.06050	0.00008	1.76765e-06	0.00178	0.06139	0.00004	1.34628e-06	0.00164
	$h(x_0)$	0.30736	0.00033	1.00478 e-06	0.00185	0.30820	0.00029	1.00469e-06	0.00200	0.30714	0.00016	1.00120e-06	0.00165